Sample Final Exam Math 235 Fall 2007

1. (*10 points*) For each statement, indicate whether the statement is true or false. FOR THIS PROBLEM BUT ONLY FOR THIS PROBLEM, no explanations are needed.

(**a**) A linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ is invertible if and only if $\ker(T) = \{0\}$.

(**b**) If $n \geq 2$ and A is an $n \times n$ matrix obtained by switching two rows of the identity matrix, then $\det A = -1$.
of $\mathbb{R}^n$, then v is in V.

(**c**) If A, B are $n \times n$ matrices and $\vec{v}$ is an eigenvector of A as well as an eigenvector of B, then $\vec{v}$ is an eigenvector of $7A - 3B$.

(**d**) If λ is an eigenvalue of a matrix A, then λ^7 is an eigenvalue of A^7.

(**e**) If $T : V \to W$ is a linear transformation, then $\dim \ker(T) + \dim \operatorname{Image}(T) = \dim V$.

(**f**) If A is an n by n matrix with $\det A = 0$, then one of the columns of A must be a scalar multiple of another column of A.

2. (*0 points*) (**a**) Using Gaussian elimination, find all solutions of the system $A\vec{x} = 0$ where A is the matrix

$$\begin{pmatrix} 2 & 2 & -1 & 12 \\ -4 & 2 & 1 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -7 & -11 \end{pmatrix}.$$

(**b**) Compute $\det A$. Use any method you wish, but **show your work**. The method "I used my calculator" will receive no points.

(**c**) Is 0 an eigenvalue for this matrix? Explain why or why not.

3. (*0 points*) Compute the characteristic equation and eigenvalues of the matrix $\begin{pmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{pmatrix}$

4. (*0 points*) Compute the projection of the vector $\vec{x} = \begin{pmatrix} 6 \\ 5 \\ 4 \\ 3 \end{pmatrix}$ onto the subpsace $V \subset \mathbb{R}^4$ spanned by $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$.

5. (*0 points*) Suppose U, V, W are vector spaces and $S : V \to V$, $T : V \to V$ are linear transformations so that the composite map $T \circ S$ is a linear transformation from V to V. Show that if S is not an isomorphism, then neither is $T \circ S$.

6. (*0 points*)

(**a**) Verify that if $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 6 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 13 \end{pmatrix}$, then $\vec{v}_1$ is not perpendicular to $\vec{v}_2$.

(**b**) Use the Gram-Schmidt process to find an orthonormal basis for the subspace V of $\mathbb{R}^4$ spanned by $\vec{v}_1, \vec{v}_2$.

7. (*0 points*)

(**a**) For an n by n martrix A, define what it means for a non-zero vector $\vec{v} \in \mathbb{R}^n$ to be an eigenvector of A and what it means for a scalar $\lambda \in \mathbb{R}$ to be an eigenvalue of A.

(**b**) Compute the eigenvalues of the matrix $A = \begin{pmatrix} -55 & 36 \\ -90 & 59 \end{pmatrix}$.

(**c**) For each eigenvalue of A, compute a corresponding eigenvector.

(**d**) Find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Describe how you arrive at S and at D. If you know what D should be but have trouble finding S, you will get some partial credit.

8. (*0 points*) Let V be the vector space of functions spanned by $\cos(2x)$ and $\sin(2x)$. Consider the map $T : V \to V$ defined by $T(f(x)) = f''(x) - f'(x)$ where $f'(x)$ is the derivative of f with respect to x.

(**a**) Using the basis $\mathcal{A} = \cos(2x), \sin(2x)$ for V, compute the matrix A which represents the linear transformation T under $\mathcal{A}$.

(**b**) Compute the determinant of A.

(**c**) Is T invertible? Why or why not?