1. (4 points) Let A be a 5×11 matrix (5 rows and 11 columns). Denote the rank of A by r.

   (a) The rank of A must be in the range ____ $\leq r \leq$ ____.

   (b) Express the dimension of the null space of A in terms of r.
   $\dim \mathrm{Null}(A) =$ _________.

   (c) Express the dimension of the column space of A in terms of r.
   $\dim \mathrm{Col}(A) =$ _________.

   (d) Express the dimension of the row space of A in terms of r.
   $\dim \mathrm{Row}(A) =$ _________.

2. (6 points) Let W be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

   (a) Find the projection of $b = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ to W.

   (b) Find the distance from b to W.

3. (18 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 7 & 8 \\ 2 & 1 & 3 & 3 & 0 & 0 \\ 3 & 2 & 4 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   a) What is the rank of A? _____

   b) Find a basis for the null space $Null(A)$ of A.

   c) Find a basis for the column space of A.

   d) Find a basis for the row space of A.

4. (18 points) Let W be the line in $\mathbb{R}^3$ spanned by $w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

   (a) Find the length of $v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

   (b) Find the projection of v to the line W.

   (c) Find the distance between v and the line W.

   (d) Denote by $W^{\perp}$ the plane (through $\vec{0}$), which is orthogonal to w. Write v as a sum of a vector in W and a vector in $W^{\perp}$.

   (e) Find the distance from v to $W^{\perp}$.

   (f) Find an orthogonal basis $\{u_1, u_2\}$ for $W^{\perp}$. *Hint:* Let u_1 be the vector in $W^{\perp}$ you found in part 4d. Now find u_2 orthogonal to both w and u_1.

5. (18 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ is $-(\lambda-1)(\lambda+1)(\lambda-2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

6. (18 points) The vectors $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .6 & .3 \\ .4 & .7 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______

The eigenvalue of v_2 is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ approaches ____. Justify your answer.

7. (18 points)

(a) Find the matrix A of the rotation of $\mathbb{R}^2$ an angle of $\frac{\pi}{2}$ radians (90°) counter-clockwise.

(b) Find the matrix B of the reflection of the plane about the line $x_1 = 0$ (the x_2 coordinate line).

(c) Compute $C = A^{-1}BA$. Is C the matrix of a rotation? (if yes, find the angle). Is C the matrix of a reflection? (if yes, find the line of reflection).

8. (18 points) Let B be the matrix $\begin{bmatrix} 4 & -7 & 4 \\ -1 & 4 & 8 \\ -8 & -4 & 1 \end{bmatrix}$ and $A = \frac{1}{9}B$.

(a) Show that the columns $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ of A form an orthonormal basis of $\mathbb{R}^3$.

(b) Use part 8a to find the coordinates of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in the basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

(c) A is the matrix of a rotation of $\mathbb{R}^3$ about a line L through the origin (you may assume this fact). **Explain** why any non-zero vector v in L must be an eigenvector of A and determine its eigenvalue.

(d) Find a vector v which spans the axis of rotation of A (the line L in part 8c). *Hint:* You may avoid calculations with fractions by working with the matrix B. Use the fact that a vector v is an eigenvector of A with eigenvalue λ, if and only if v is an eigenvector of B with eigenvalue 9λ.