1. (4 points) Let $A$ be a $5 \times 11$ matrix (5 rows and 11 columns). Denote the rank of $A$ by $r$.
   (a) The rank of $A$ must be in the range $\leq r \leq \underline{\hspace{1cm}}$.
   (b) Express the dimension of the null space of $A$ in terms of $r$.
      \[ \dim \text{Null}(A) = \underline{\hspace{1cm}}. \]
   (c) Express the dimension of the column space of $A$ in terms of $r$.
      \[ \dim \text{Col}(A) = \underline{\hspace{1cm}}. \]
   (d) Express the dimension of the row space of $A$ in terms of $r$.
      \[ \dim \text{Row}(A) = \underline{\hspace{1cm}}. \]

2. (6 points) Let $W$ be the plane in $\mathbb{R}^3$ spanned by
   \[ u_1 = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} \] and
   \[ u_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \]
   (a) Find the projection of \( b = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \) to $W$.
   (b) Find the distance from $b$ to $W$.

3. (18 points) The matrices $A$ and $B$ below are row equivalent (you do not need to check this fact).
   \[
   A = \begin{pmatrix}
   1 & 1 & 1 & 2 & 7 & 8 \\
   2 & 1 & 3 & 3 & 0 & 0 \\
   3 & 2 & 4 & 5 & 1 & 4 \\
   0 & 0 & 0 & 0 & 3 & 2 \\
   0 & 0 & 0 & 0 & 0 & 1 \\
   \end{pmatrix}
   \]
   \[
   B = \begin{pmatrix}
   1 & 0 & 2 & 1 & 0 & 0 \\
   0 & 1 & -1 & 1 & 0 & 0 \\
   0 & 0 & 0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0 & 0 & 0 \\
   \end{pmatrix}
   \]
   a) What is the rank of $A$?
   b) Find a basis for the null space $\text{Null}(A)$ of $A$.
   c) Find a basis for the column space of $A$.
   d) Find a basis for the row space of $A$.

4. (18 points) Let $W$ be the line in $\mathbb{R}^3$ spanned by $w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
   (a) Find the length of $v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.
   (b) Find the projection of $v$ to the line $W$.
   (c) Find the distance between $v$ and the line $W$.
   (d) Denote by $W^\perp$ the plane (through $\vec{0}$), which is orthogonal to $w$. Write $v$ as a sum of a vector in $W$ and a vector in $W^\perp$.
   (e) Find the distance from $v$ to $W^\perp$.
   (f) Find an orthogonal basis \{\(u_1, u_2\)\} for $W^\perp$. \textit{Hint:} Let $u_1$ be the vector in $W^\perp$ you found in part 4d. Now find $u_2$ orthogonal to both $w$ and $u_1$.

5. (18 points)
(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ is $-(\lambda - 1)(\lambda + 1)(\lambda - 2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

(c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that the matrix $A$ above satisfies $P^{-1}AP = D$.

6. (18 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .6 & .3 \\ .4 & .7 \end{pmatrix}$.

(a) The eigenvalue of $v_1$ is _______

   The eigenvalue of $v_2$ is _______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(d) As $n$ gets larger, the vector $A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ approaches _______. Justify your answer.

7. (18 points)

(a) Find the matrix $A$ of the rotation of $\mathbb{R}^2$ an angle of $\frac{\pi}{2}$ radians ($90^\circ$) counter-clockwise.

(b) Find the matrix $B$ of the reflection of the plane about the line $x_1 = 0$ (the $x_2$ coordinate line).

(c) Compute $C = A^{-1}BA$. Is $C$ the matrix of a rotation? (if yes, find the angle). Is $C$ the matrix of a reflection? (if yes, find the line of reflection).

8. (18 points) Let $B$ be the matrix $\begin{pmatrix} 4 & -7 & 4 \\ -1 & 4 & 8 \\ -8 & -4 & 1 \end{pmatrix}$ and $A = \frac{1}{7}B$.

(a) Show that the columns $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ of $A$ form an orthonormal basis of $\mathbb{R}^3$.

(b) Use part 8a to find the coordinates of the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in the basis $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

(c) $A$ is the matrix of a rotation of $\mathbb{R}^3$ about a line $L$ through the origin (you may assume this fact). **Explain** why any non-zero vector $v$ in $L$ must be an eigenvector of $A$ and determine its eigenvalue.

(d) Find a vector $v$ which spans the axis of rotation of $A$ (the line $L$ in part 8c). **Hint:** You may avoid calculations with fractions by working with the matrix $B$. Use the fact that a vector $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, if and only if $v$ is an eigenvector of $B$ with eigenvalue $9\lambda$. 

2