Math 235H Midterm 2 Fall 2018

Your Name: Solution

Student ID: ____________________

This is a 90 minutes exam. This exam paper consists of 5 questions. It has 8 pages.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (20)	________
2. (20)	________
3. (20)	________
4. (20)	________
5. (20)	________
TOTAL (100)	

Free variables

$x_1\ x_2\ (x_3)\ x_4\ (x_5\ x_6)$

1. (20 points) You are given below the matrix A together with its row reduced echelon form B (you need *not* verify that B is indeed row equivalent to A).

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 1 & 3 & 2 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 2 & -1 & 1 & 4 & 3 & 1 \end{pmatrix}. \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = -x_3 + x_5 + 2x_6$
$x_2 = -x_3 + x_5 + x_6$
$x_4 = -x_5 - x_6$

7 pt

a) Find a basis for the null space $Null(A)$ of A. Justify your answer!

The general sol'n of $A\vec{x} = \vec{0}$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -x_3 + x_5 + 2x_6 \\ -x_3 + x_5 + x_6 \\ x_3 \\ -x_5 - x_6 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_5 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + x_6 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$$

$Null(A)$ = Set of sol'n of $A\vec{x} = \vec{0}$ is spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, as shown above. Furthermore, if $x_3\vec{v}_1 + x_5\vec{v}_2 + x_6\vec{v}_3 = \vec{0}$, then $x_3 = 0$, $x_5 = 0$, $x_6 = 0$, being the 3-rd, 5-th, and 6-th entries of the linear combination.

7 pts

b) Find a basis for the column space $Col(A)$ of A. Justify your answer!

The pivot columns $\vec{a}_1, \vec{a}_2, \vec{a}_4$ of A form a basis for $Col(A)$.

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{a}_4 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \end{pmatrix}.$$

(6 pt)

c) Is the sixth column $\vec{a}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ of the matrix A in part 1a) a linear combination of the first five columns of A? Justify your answer avoiding any new computations.

Yes, $\vec{a}_6$ belongs to Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5, \vec{a}_6\} = \text{Col}(A)$ an so it is a linear combination of the pivot columns $\vec{a}_1, \vec{a}_2$, and $\vec{a}_4$. If $\vec{a}_6 = c_1\vec{a}_1 + c_2\vec{a}_2 + c_4\vec{a}_4$, then $\vec{a}_6 = c_1\vec{a}_1 + c_2\vec{a}_2 + 0\cdot\vec{a}_3 + c_4\vec{a}_4 + 0\vec{a}_5$, so it is a linear combination of the first 5 columns of A.

Note: $\vec{b}_6 = -2\vec{b}_1 - \vec{b}_2 + \vec{b}_4$. So $\vec{a}_6 = -2\vec{a}_1 - \vec{a}_2 + \vec{a}_4$, since The linear relations among the columns of A and B are the same.

2. (a) (10 points) The determinant of the 3×3 matrix $A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, with rows a_1, a_2, and a_3, is 5. Compute $\det \begin{pmatrix} 2a_1 \\ 3a_1 + 3a_2 \\ 4a_1 + 2a_3 \end{pmatrix}$.

$$\det\begin{pmatrix} 2a_1 \\ 3a_1+3a_2 \\ 4a_1+2a_3 \end{pmatrix} = 2\det\begin{pmatrix} a_1 \\ 3a_1+3a_2 \\ 4a_1+2a_3 \end{pmatrix} = 2\det\begin{pmatrix} a_1 \\ 3a_2 \\ 2a_3 \end{pmatrix} =$$

Add $-3R_1$ to R_2, Add $-4R_1$ to R_3 } does not change det

$$= 2\cdot 3\cdot 2\det\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 60.$$

(where $\det\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 5$)

(b) (10 points) Let A, B, and C be invertible $n \times n$ matrices. Show that there exists precisely one $n \times n$ matrix X satisfying $A(B+X)C = B$. Express X in terms of A, B, and C. Multibly both sides by A^{-1} on the left and C^{-1} on the right to get: $B + X = A^{-1}BC^{-1}$.

Subtract B from both sides to get

$$X = A^{-1}BC^{-1} - B$$

3. (20 points) Let $\vec{u}$ be a non-zero vector in $\mathbb{R}^2$. Recall that the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane with respect to the line $L = \text{span}\{\vec{u}\}$ is given by the formula $R(\vec{x}) = 2\left(\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}\right)\vec{u} - \vec{x}$. Note that vectors in $\mathbb{R}^2$ are considered here as 2×1 matrices, and we regard the 1×1 matrices $\vec{x}^T\vec{u}$ and $\vec{u}^T\vec{u}$ as scalars, so that the fraction $\frac{\vec{x}^T\vec{u}}{\vec{u}^T\vec{u}}$ is a scalar (quotient of two dot products).

(a) Show that R is a linear transformation by verifying the two properties in the definition of a linear transformation.

(1) Let v, w be vectors in $\mathbb{R}^2$. Note that $(v+w)^T\vec{u} = v^Tu + w^Tu$. So

$$R(v+w) = 2\left(\frac{v^Tu + w^Tu}{u^Tu}\right)u - (v+w) = \left[2\left(\frac{v^Tu}{u^Tu}\right)u - v\right] + \left[2\left(\frac{w^Tu}{u^Tu}\right)u - w\right] =$$

$$= R(v) + R(w).$$

(2) Let $\vec{v}$ be a vector in $\mathbb{R}^2$ and c a scalar. Note that $(c\vec{v})^T\vec{u} = c(\vec{v}^T\vec{u})$. So

$$R(c\vec{v}) = 2c\left(\frac{\vec{v}^Tu}{u^Tu}\right)u - c\vec{v} = c\left[2\left(\frac{v^Tu}{u^Tu}\right)u - \vec{v}\right] = cR(\vec{v})!$$

(10 pt)

(b) Assume (only) in this part that $\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $L = \text{span}\{\vec{u}\}$ as above. Find the standard matrix A of the reflection $R : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to L.

$$\overset{\shortparallel}{(\vec{a}_1\ \vec{a}_2)}$$

$$\vec{a}_1 = A\begin{pmatrix}1\\0\end{pmatrix} \overset{\text{WANT}}{=} R\begin{pmatrix}1\\0\end{pmatrix} = 2\underbrace{\left(\frac{(1\ 0)\begin{pmatrix}1\\3\end{pmatrix}}{(1\ 3)\begin{pmatrix}1\\3\end{pmatrix}}\right)}_{1/10}\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-4\\3\end{pmatrix}$$

$$\vec{a}_2 = A\begin{pmatrix}0\\1\end{pmatrix} \overset{\text{WANT}}{=} R\begin{pmatrix}0\\1\end{pmatrix} = 2\underbrace{\left(\frac{(0\ 1)\begin{pmatrix}1\\3\end{pmatrix}}{(1\ 3)\begin{pmatrix}1\\3\end{pmatrix}}\right)}_{3/10}\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{3}{5}\begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$$

So $\boxed{A = \frac{1}{5}\begin{pmatrix}-4 & 3\\ 3 & 4\end{pmatrix}}$

10 pts

4. (20 points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}$. a) Compute A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -2 & 5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

(right block: A^{-1})

10 pts

b) Let $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Note that $B^{-1} = B$. Compute $(AB)^{-1}$ with as few computations as possible. Hint: Do not compute AB.

$$(AB)^{-1} = B^{-1}A^{-1} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{B=B^{-}} \underbrace{\begin{pmatrix} -4 & -2 & 5 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}}_{A^{-1}} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ -4 & -2 & 5 \end{pmatrix}$$

5. (20 points) Let P_3 be the vector space of polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ of degree less than or equal to 3, with real coefficients. Let $T : P_3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(p) := \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}. \tag{1}$$

(a) Is the subset $\{T(1), T(x), T(x^2), T(x^3)\}$ of $\mathbb{R}^3$ linearly dependent or independent?

Linearly dependent, since more vectors than entries in each vector (when placed as columns of a matrix we get a 3×4 matrix that has at most 3 pivot, so can not have a pivot in every column.

(b) Show that the image im(T) of T is the whole of $\mathbb{R}^3$. Recall that im(T) is the set of values of T.

$T(a_0 + a_1x + \ + a_m x^3) = a_0 T(1) + a_1 T(x) + a_2 T(x^2) + a_3 T(x^3)$, so

$\mathrm{Im}(T) = \mathrm{Span}\{T(1), T(x), T(x^2), T(x^3)\} =$

$\mathrm{Span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\4\end{pmatrix}, \begin{pmatrix}0\\1\\8\end{pmatrix} \right\}$. It suffices to show that the matrix

$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{pmatrix}$ has a pivot in every row.

$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \end{pmatrix}$. Indeed, we have a pivot in every row.

(c) Let $\ker(T)$ be the set $\{\vec{x}$, such that $T(\vec{x}) = \vec{0}\}$. Show that $\ker(T)$ is a subspace by verifying the conditions in the definition of a subspace. *Hint: The notation would be simpler if you show it more generally for every linear transformation $T : V \to W$ between vector spaces V and W.* The three conditions are:

(1) $\vec{0}$ is in $\ker(T)$. Indeed $\vec{0} = \vec{0} + \vec{0}$, so,
$T(\vec{0}) = T(\vec{0}+\vec{0}) = T(\vec{0}) + T(\vec{0})$ (← Linearity of T). Adding $-T(\vec{0})$ to both sides we get $T(\vec{0}) = \vec{0}$.

(2) Let $\vec{u}$ and $\vec{v}$ be vectors in $\ker(T)$. Then
$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}$ (↑ Linearity of T). So $\vec{u}+\vec{v}$ is in $\ker(T)$.

(3) Let $\vec{v}$ be in $\ker(T)$ and c a scalar. Then
$T(c\vec{v}) = cT(\vec{v}) = c \cdot \vec{0} = \vec{0}$ (↑ Linearity of T). So $c\vec{v}$ belongs to $\ker(T)$.

(d) Find a basis for $\ker(T)$, where T is the linear transformation in Equation (1). Justify your answer!

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = T(a_0 + a_1x + a_2x^2 + a_3x^3) = \underbrace{\begin{pmatrix}1&0&0&0\\1&1&1&1\\1&2&4&8\end{pmatrix}}_{A}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix}$$

$$= a_0T(1) + a_1T(x) + a_2T(x^2) + a_3T(x^3)$$

$$A \sim \begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&2&6\end{pmatrix} \sim \begin{pmatrix}1&0&0&0\\0&1&0&-2\\0&0&1&3\end{pmatrix}.$$

(↑ Part b)

$a_0 = 0$, $a_1 = 2a_3$, $a_2 = -3a_3$, $a_3 = a_3$

$$\begin{pmatrix}a_0\\a_1\\a_2\\a_3\end{pmatrix} = a_3\underbrace{\begin{pmatrix}0\\2\\-3\\1\end{pmatrix}}_{\vec{v}}$$

Choose $a_3 = 1$. Let $\mathcal{B}$ be the basis $\{1, x, x^2, x^3\}$ of $\mathbb{P}_3$. We get that if a polynomial p belongs to $\ker(T)$, then the coordinate vector $[p]_{\mathcal{B}}$ is a scalar multiple of $\vec{v}$. So p is a scalar multiple of

$f(x) = 2x - 3x^2 + x^3 = x(2 - 3x + x^2) = x(x-1)(x-2)$.

Hence f spans $\ker(T)$ and is a basis for T.