Math 235H Midterm 1 Fall 2018

Your Name: Solution

Student ID: ______________________

This is a 90 minutes exam. This exam paper consists of 5 questions. It has 8 pages.

The use of calculators is not allowed on this exam. You may use one letter size page of notes (both sides), but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (20)	______
2. (20)	______
3. (20)	______
4. (20)	______
5. (20)	______
TOTAL (100)	

1. (20 points) a) Show that the row **reduced** echelon augmented matrix of the system

$$\begin{aligned} x_1 - x_2 + x_4 + x_5 &= 0 \\ 2x_1 - x_2 + x_3 + 3x_4 + 2x_5 &= 0 \\ -x_1 + x_2 &= 1 \end{aligned} \quad \text{is} \quad \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Use at most five elementary row operations. Clearly write in words each elementary row operation you use.

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 1 & 0 \\ 2 & -1 & 1 & 3 & 2 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$ Add $-2R_1$ to R_2

$\sim$ Add R_1 to R_3

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$ Add $-R_3$ to R_1

$\sim$ " " " R_2

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$\sim$ Add R_2 to R_1

$$\begin{array}{cccccc} & & x_3 & & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

5 pt

b) Find the general solution for the system in parametric form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 - x_3 + x_5 \\ -1 - x_3 + x_5 \\ x_3 \\ 1 - x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$A\vec{x} = \vec{0}$

c) Find the general solution of the associated homogeneous system (replacing the constant, on the right hand side of the equations in part a, by zeros). Try to avoid computations. Justify your answer!

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_5 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{v_2}.$$

Reason. Row reduction of the augmented matrix $(A|\vec{0})$ of the homogeneous system will result in replacing the rightmost column of the original reduced echelon form by the zero column, and so the constant particular solution $\vec{p} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ in part b will be replaced by the zero vector here.

d) Find two vectors $\{v_1, v_2\}$, such that the solution set in Part c is span$\{v_1, v_2\}$. Justify your answer starting from the definition of span$\{v_1, v_2\}$.

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

In part (c) we see that the general solution of the associated homogeneous system is an arbitrary linear combination of these two vectors. This is precisely the general element of Span$\{v_1, v_2\}$, since

$$\underbrace{\text{Span}\{\vec{v}_1, \vec{v}_2\} = }_{\text{Definition}} \text{Set of all linear combinations } c_1\vec{v}_1 + c_2\vec{v}_2 \text{ of } \vec{v}_1 \text{ and } \vec{v}_2.$$

2. (20 points) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 3 \\ 2 \\ h \end{bmatrix}$. $A = (v_1 v_2 v_3)$

a) For which real numbers h is the vector $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ in span$\{v_1, v_2, v_3\}$? Justify your answer!

Add $-2R_1$ to R_2 Add $-2R_2$ to R_3

$$(A|\vec{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 2 & -1 \\ 3 & 1 & h & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -4 & -3 \\ 0 & -2 & (h-9) & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -1 & -4 & -3 \\ 0 & 0 & (h-1) & 3 \end{array}\right)$$

Add $-3R_1$ to R_3

If $h=1$ we have a pivot in the rightmost column and so the system $A\vec{x} = \vec{b}$ is inconsistent. Hence, $\vec{b}$ is not a linear combination of the columns v_1, v_2, v_3 of A.
If $h \neq 1$, then A has a pivot in every row, and so $A\vec{x} = \vec{b}$ is consistent for every vector $\vec{b}$, and in particular for $\vec{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

b) For which values of h are the three vectors $\{v_1, v_2, v_3\}$ linearly independent? Justify your answer without additional computations.

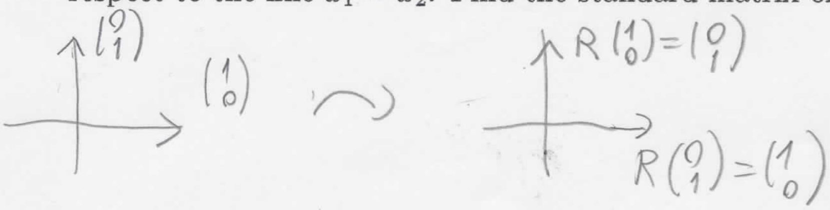
If $h \neq 1$ we have a pivot position in every column of the matrix $(v_1 v_2 v_3)$, and so its columns are linearly independent.

If $h=1$, we do not have a pivot position in the 3-rd column of $(\vec{v}_1 \vec{v}_2 \vec{v}_3)$, and so the three vectors are linearly dependent.

3. (20 points)

7 pts

(a) Let $R : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation reflecting each vector with respect to the line $x_1 = x_2$. Find the standard matrix of R.



Standard matrix of R is

$$A = \left(R\left(\begin{pmatrix}1\\0\end{pmatrix}\right) \; R\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \right) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

7 pts

(b) Find the standard matrix of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by $S(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2)$.

$S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

$S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

standard matrix is

$$B = \left(S\left(\begin{pmatrix}1\\0\end{pmatrix}\right) S\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \right) \qquad B = \begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}$$

6 pts

(c) Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by $T(\vec{x}) = S(R(\vec{x}))$. Show all your work!

standard matrix of T is $BA = \begin{pmatrix}2 & 3\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}3 & 2\\ -1 & 1\end{pmatrix}$

check: $T\begin{pmatrix}1\\0\end{pmatrix} = S\left(R\begin{pmatrix}1\\0\end{pmatrix}\right) = S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = S\left(R\begin{pmatrix}0\\1\end{pmatrix}\right) = S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$

4. (20 points) Determine if the statement is true or false. Justify your answer! (credit will be given only if a valid justification is provided).

7 pts

(a) If A is a $n \times n$ matrix and there is a vector $\vec{b}$ in $\mathbb{R}^n$, such that the equation $A\vec{x} = \vec{b}$ is inconsistent, then the equation $A\vec{x} = \vec{0}$ has a non-zero solution.

True, $A\vec{x} = \vec{b}$ is inconsistent for some $\vec{b}$ $\Rightarrow$

A does not have a pivot position in every row $\Rightarrow$

A has less than n pivot positions $\Rightarrow$

A does not have a pivot pos in every column $\Rightarrow$

$A\vec{x} = \vec{0}$ has a non-zero sol'n,

7 pts

(b) If the set $\{v_1, v_2, v_3\}$ of three vectors in $\mathbb{R}^5$ is linearly dependent, then so is the set $\{T(v_1), T(v_2), T(v_3)\}$, for every linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^6$.

True. Assume $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, and not all c_i are zero. Then

$\vec{0} = T(\vec{0}) = T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + c_3 T(\vec{v}_3).$

Linearly properties of T

Hence, $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is lin. dep.

(c) If A is a $m \times n$ matrix and $n > m$, then the matrix equation $A\vec{x} = \vec{b}$ has infinitely many solutions for every vector $\vec{b}$ in $\mathbb{R}^m$.

False. $A\vec{x} = \vec{b}$ may be inconsistent for some $\vec{b}$.

Take for example $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

5. (20 points) Let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ a) Find the general form of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which satisfies $AB = BA$. Justify your answer.

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$AB = BA \Rightarrow c = b,\ d = a,\ a = d,\ b = c,$

So, the general form is $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$,

where a, b are arbitrary.

b) Let A and B be square $n \times n$ matrices, such that the columns of B are linearly independent. Assume that the equation $(AB)\vec{x} = \vec{0}$ has a non-zero solution $\vec{x}$. Show that the columns of A are linearly dependent. Hint: $(AB)\vec{x} = A(B\vec{x})$).

Let $\vec{x}$ be a non-zero vector, such that $(AB)\vec{x} = \vec{0}$. Then $\vec{y} = B\vec{x} \neq \vec{0}$, since the columns of B are linearly independent.

Now $A\vec{y} = A(B\vec{x}) = (AB)\vec{x} = \vec{0}$.

given

Hence, the system $A\vec{y} = \vec{0}$ has a non-zero solution $\vec{y} = B\vec{x}$. Thus, the columns of A are linearly dependent.