Math 235 Midterm 2 Fall 2015

1. (20 points) You are given below the matrix A together with its row reduced echelon form B (you need *not* verify that B is indeed the reduced echelon form of A)

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for the null space $Null(A)$ of A. Justify!

b) Find a basis for the column space $Col(A)$ of A. Justify!

c) Is the sixth column $a_6 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ of the matrix A in part 1a) a linear combination of the first five columns of A? Justify your answer. Hint: A careful reading of Question 1 will eliminate the need for any computations.

2. (a) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2) = (3x_1 - 4x_2, -5x_1 + 7x_2).$$

Show that T is invertible and find a formula for T^{-1}.

(b) (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Recall that a vector in $\mathbb{P}_2$ is a polynomial $p(x)$ of the form $p(x) = a_0 + a_1 x + a_2 x^2$, where the coefficients a_0, a_1, a_2 are arbitrary real numbers. Is the subset $\{f, g, h\}$ of $\mathbb{P}_2$, consisting of the three polynomials $f(x) = 1 - x$, $g(x) = 1 - x^2$, and $h(x) = 1 + x + x^2$, linearly dependent or independent? Justify your answer.

3. a) (8 points) Let A, B, and C be invertible $n \times n$ matrices. Show that there exists precisely one $n \times n$ matrix X satisfying $C(A + X)B = A$. Express X in terms of A, B, and C.

b) (12 points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Compute its inverse A^{-1}. (**Check** that $AA^{-1} = I$.)

4. (20 points) Determine if the following subset H of $\mathbb{R}^n$ is a subspace. If it is not, find a property in the definition of a subspace which H violates. If H is a subspace find either a set of vectors which spans it, or a matrix A such that H is $Null(A)$ (which will provide the justification that it is indeed a subspace).

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } xy \geq 0 \right\}$ (the union of the first and third quadrants).

(b) $H = \left\{ \begin{bmatrix} 4x_1 + x_3 \\ 2x_1 - 3x_2 \\ x_2 + 6x_3 \end{bmatrix} \text{ such that } x_1,\ x_2,\ x_3 \text{ are arbitrary real numbers} \right\}$

5. (20 points) a) Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $\vec{0}$, v_1, v_2, v_3, v_1+v_2, v_1+v_3, v_2+v_3, $v_1+v_2+v_3$ (the parallelepiped determined by v_1, v_2, and v_3) where

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

b) Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending a vector $\vec{x}$ to $A\vec{x}$, where A is the matrix $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Suppose v_1, v_2 are two vectors in $\mathbb{R}^2$, such that the parallelogram with vertices 0, v_1, v_2, v_1+v_2 has area 8 square meters. Compute the area of the image of this parallelogram under the transformation T. (The image is the parallelogram with vertices 0, $A(v_1)$, $A(v_2)$, and $A(v_1+v_2)$). **Justify your answer!**