Math 235 Midterm 2 Fall 2006

1. (24 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 2 & 2 & 2 \\ 2 & 1 & 4 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Note: you do **not** have to check that A and B are indeed row equivalent.

   a) Find a basis for the null space $Null(A)$ of A.

   b) Find a basis for the column space of A.

   c) Is the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in the column space of A? Use part b to **Justify** your answer!

2. (16 points) a) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$. Find a matrix B satisfying $AB = C$.

   b) Check your answer in part a by calculating AB.

3. a) (6 points) Let $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, and B a 2×2 matrix satisfying the equation

$$A = PBP^{-1}. \tag{1}$$

   Calculate the matrix B. *Hint: First express B in terms of A and P.* (**Check** that your B satisfies equation (1).)

   b) (10 points) Let $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Compute its inverse A^{-1}. (**Check** that $AA^{-1} = I$.)

4. (18 points) Determine which of the following sets in $\mathbb{R}^n$ is a subspace. If it is not, **find** a property in the definition of a subspace which this set violates. If it is a subspace, **find** a matrix A such that this set is either $Null(A)$ or $Col(A)$.

   (a) $\left\{ \begin{bmatrix} a+3b-2 \\ a+b \\ a-b \end{bmatrix} \text{ such that } a, b \text{ are arbitrary real numbers} \right\}$

   (b) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1, x_2, x_3, x_4 \text{ are real numbers satisfying } \begin{array}{rcl} x_1 + x_2 + x_3 & = & x_4 \\ 5x_2 & = & 4x_3 \end{array} \right\}$

(c) $\left\{ \begin{bmatrix} x_1 + 5x_3 \\ -2x_1 + 2x_2 \\ x_2 + 3x_3 \\ x_1 - x_3 \end{bmatrix} \text{ such that } x_1,\ x_2,\ x_3 \text{ are arbitrary real numbers} \right\}$

5. (16 points) a) Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $\vec{0}$, v_1, v_2, v_3, $v_1 + v_2$, $v_1 + v_3$, $v_2 + v_3$, $v_1 + v_2 + v_3$, where

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

b) Use your answer in part (b) and the algebraic properties of determinants to compute the volume of the parallelepiped obtained if v_1, v_2, v_3 are replaced by w_1, w_2, w_3, where $w_i = 2v_i$, for $i = 1, 2, 3$.

6. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Recall that a vector in $\mathbb{P}_2$ is a polynomial $p(t)$ of the form $p(t) = a_0 + a_1 t + a_2 t^2$, where the coefficients a_0, a_1, a_2 are arbitrary real numbers.

   (a) Find a polynomial $p(t)$, of degree at most 2, satisfying $p(0) = 4$, $p(1) = 1$, and $p(2) = 0$.

   (b) The subset H of $\mathbb{P}_2$, of polynomials $p(t)$ of degree ≤ 2, which in addition satisfy

   $$p(2) = 0$$

   is a *subspace* of $\mathbb{P}_2$. (You may assume this fact). Find a basis for H. **Explain** why the set you found is linearly independent and why it spans H.