Math 235 Final Exam Fall 2001

1. (16 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for the null space $Null(A)$ of A.

b) Find a basis for the column space of A.

c) Find a basis for the row space of A.

2. (16 points)

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 6 & 0 & -4 \\ 0 & 1 & 0 \\ 8 & 0 & -6 \end{pmatrix}$ is

$-(\lambda - 1)(\lambda + 2)(\lambda - 2)$.

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

3. (4 point) Let A be a 6×10 matrix (6 rows and 10 columns). Denote the dimension of the column space of A by r.

(a) The dimension r of the column space must be in the range
$\_\_\_\_ \leq r \leq \_\_\_\_$.

(b) Express the dimension of the null space of A in terms of r.
$\dim \mathrm{Null}(A) = \_\_\_\_\_\_\_\_\_$.

(c) Express the dimension of the row space of A in terms of r.
$\dim \mathrm{Row}(A) = \_\_\_\_\_\_\_\_\_$.

4. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______

The eigenvalue of v_2 is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches _____. Justify your answer.

5. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -7 \\ 4 \\ -4 \end{bmatrix}$.

(a) Find the projection $\mathrm{Proj}_W(v)$ of $v = \begin{bmatrix} 8 \\ 7 \\ -7 \end{bmatrix}$ to W.

(b) Find the distance from v to W.

(c) Set $u_3 := v - \mathrm{Proj}_W(v)$ and let U be the 3×3 matrix with columns u_1, u_2, and u_3. Show that $\frac{1}{9}U$ is an **orthogonal** matrix.

(d) Find the distance, from the vector $\left(\frac{1}{9}U^T\right)v$ to the plane in $\mathbb{R}^3$ spanned by the vactors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, without any further calculations. Explain your answer!
Hint: where does $\frac{1}{9}U$ take the three vectors above?

(e) $\frac{1}{9}U$ is the matrix of a rotation of $\mathbb{R}^3$ about a line L through the origin (you may assume this fact). Find a vector w which spans the line L (the axis of rotation).

6. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

(a) Find the projection of a_2 to the line spanned by a_1.

(b) Find the distance from a_2 to the line spanned by a_1.

(c) Use your calculations in parts 6a and 6b to show that the vectors $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ form an orthogonal basis of the plane W given above.

(d) Find the projection of $b = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ to W.

(e) Find a least square solution of the equation $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the 3×2 matrix with columns a_1 and a_2. I.e., find a vector x in $\mathbb{R}^2$, for which the distance $\|Ax - b\|$ from Ax to b is minimal.

7. (16 points)

(a) Find the matrix A of the rotation of $\mathbb{R}^2$ an angle of $\frac{\pi}{4}$ radians (45°) counter-clockwise.

(b) Find the matrix B of the reflection of the plane about the line $x_2 = 0$ (the x_1 coordinate line).

(c) Compute $C = B^{-1}AB$.

(d) Show that C is the matrix of a rotation and find the angle of rotation.