Math 235 section 4 Solution of Midterm 1 Fall 2008

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system
$x_1 + x_3 - x_4 - 2x_5 = 2$
$x_1 + x_2 + 3x_3 = 1$
$2x_1 + 2x_3 + x_4 + 5x_5 = 1$
is $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -1 \end{pmatrix}$. Use at most five elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

Answer: Add $-R_1$ to R_2. Add $-2R_1$ to R_3. Multiply R_3 by 1/3. Add $-R_3$ to R_2. Add R_3 to R_1.

b) Find the general solution for the system.

Answer: The free variables are x_3 and x_5.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

2. (20 points) You are given that the row reduced echelon form of the matrix
$A = \begin{pmatrix} 3 & 6 & 1 & 2 & 6 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 2 & 2 & 0 & -1 & 1 \end{pmatrix}$ is $B = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. You do **not** need to verify this statement.

(a) Write the general solutions of the system $A\vec{x} = \vec{0}$ in parametric form
$\vec{x} =$ (first free variable)$\vec{v}_1$ + (second free variable)$\vec{v}_2 + \ldots$

Answer: The free variables are x_2, x_5, and x_6.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_5 + x_6 \\ x_2 \\ x_5 - x_6 \\ -2x_5 + x_6 \\ x_5 \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(b) Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear transformations given by $T(\vec{x}) = A\vec{x}$. Find a *finite* set of vectors in $\ker(T)$, which spans $\ker(T)$. Explain why the set you found spans $\ker(T)$.

Answer: Note: The word "finite" was missing in the exam version, so other solutions were given full credit.
The kernel $\ker(T)$ is the set of solutions of the equation $A\vec{x} = \vec{0}$. We wrote the general solution $\vec{x}$ as a linear combination $\vec{x} = x_2\vec{v}_1 + x_5\vec{v}_2 + x_6\vec{v}_3$, for the three vectors $\vec{v}_1$,$\vec{v}_2$,$\vec{v}_3$ given on the right hand side of the answer to part 2a. Hence, $\ker(T)$ is the set of all linear combinations of $\vec{v}_1$,$\vec{v}_2$,$\vec{v}_3$. The latter

is, by definition span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. We conclude that the set of three vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\ker(T)$.

(c) Let $\vec{a}_j$ be the j-th column of A. Explain, without any further calculations, why $\vec{a}_6$ belongs to span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$.

Answer: Regard the matrix A as the augmented matrix $(C|\vec{a}_6)$, where $C = (\vec{a}_1\vec{a}_2\vec{a}_3\vec{a}_4\vec{a}_5)$ is the 4×5 coefficient matrix of a linear system $C\vec{x} = \vec{a}_6$. Since the row reduced echelon form B does not have a pivot in the rightmost column, then the system $C\vec{x} = \vec{a}_6$ is consistent. Hence $\vec{a}_6$ is a linear combination of the columns of C.

(d) Is the image of T equal to the whole of $\mathbb{R}^4$? Justify your answer.

Answer: No, since we do not have a pivot position in every row.

3. (a) (10 points) Determine for which values of k the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & k+1 \end{pmatrix}$ is invertible, and find the inverse when it exists.

Answer: We row reduce $(A|I)$ attempting to get $(I|A^{-1})$. The values of k for which we succeed below are those for which the matrix is invertible.
Add $-R_1$ to R_2 and add $-R_1$ to R_3. Then add -3 times R_2 to R_3 to get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 4 & k+1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & k & 2 & -3 & 1 \end{pmatrix}.$$

We see that the matrix is invertible for all values of k except $k = 0$. Assuming $k \neq 0$, divide R_3 by k, subtract R_3 from R_1, and subtract R_2 from R_1 to get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & k & 2 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2-\frac{2}{k} & -1+\frac{3}{k} & \frac{-1}{k} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{k} & \frac{-3}{k} & \frac{1}{k} \end{pmatrix}.$$

(b) (2 points) Check that the matrix you found is indeed A^{-1}.

Answer: $AA^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & k+1 \end{pmatrix} \begin{pmatrix} 2-\frac{2}{k} & -1+\frac{3}{k} & \frac{-1}{k} \\ -1 & 1 & 0 \\ \frac{2}{k} & \frac{-3}{k} & \frac{1}{k} \end{pmatrix} = I.$

(c) (8 points) Let A, B, C be $n \times n$ matrices, with A invertible, which satisfy the equation $ACA^{-1} - A = B$. Express C in terms of A and B.

Answer: Add A to both sides to get $ACA^{-1} = A + B$. Multiply both sides by A^{-1} on the left to get $CA^{-1} = A^{-1}(A+B)$. Multiply both sides by A on the right to get $C = A^{-1}(A+B)A = A + A^{-1}BA$.

4. (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with standard matrix A. Assume that the image of T is the whole of $\mathbb{R}^2$. Carfully **justify** your answers to the following questions.

(a) The rank of A is: $\underline{\ \ 2\ \ }$.

Reason: The image $im(T)$ is the whole of $\mathbb{R}^2$, if and only if A has a pivot in every row. A is a 2×3 matrix, so A has two pivot positions.

(b) Describe geometrically the kernel of T.

Answer: The kernel $\ker(T)$ is the set of solutions of $A\vec{x} = \vec{0}$. A has two pivots, so the system has precisely *one* free variable x_i. The set of solutions thus has the parametric form $x_i\vec{v}$, for some non-zero vector $\vec{v}$. Thus $\ker(T)$ is the LINE in $\mathbb{R}^3$ spanned by $\vec{v}$, i.e., the line through $\vec{0}$ and $\vec{v}$.

(c) Consider the standard matrix A of T. Fix one solution $\vec{p}$ of the equation

$$A\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{1}$$

i. Show that if a vector $\vec{x}$ is a solution of the above equation, then $\vec{x} - \vec{p}$ is in the kernel of T.

Answer: Assume that $\vec{x}$ is a solution of the above equation. Then $T(\vec{x} - \vec{p}) = A(\vec{x} - \vec{p}) = A\vec{x} - A\vec{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{0}$.

ii. Conversely, show that if $\vec{x} - \vec{p}$ is in the kernel of T, then $\vec{x}$ is a solution of equation (1).

Answer: Assume that $\vec{x} - \vec{p}$ belongs to the kernel of T. Then
$\vec{0} = T(\vec{x} - \vec{p}) = A(\vec{x} - \vec{p}) = A\vec{x} - A\vec{p}$. So $A\vec{x} = A\vec{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

5. (25 points) Let L be the line in $\mathbb{R}^2$ through the origin and the non-zero vector $\vec{u}$. Recall that the projection $Proj_L : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane onto the line L is given by the formula $Proj_L(\vec{x}) = \left(\frac{\vec{u}\cdot\vec{x}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

(a) Use the algebraic properties of the dot product to show that $Proj_L$ is a linear transformation. In other words, verify the following identities, for any two vectors $\vec{v}, \vec{w}$ and for every scalar k.

i. $Proj_L(\vec{v} + \vec{w}) = Proj_L(\vec{v}) + Proj_L(\vec{w})$.

Answer: $Proj_L(\vec{v} + \vec{w}) = \left(\frac{\vec{u}\cdot(\vec{v}+\vec{w})}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{(\vec{u}\cdot\vec{v})+(\vec{u}\cdot\vec{w})}{\vec{u}\cdot\vec{u}}\right)\vec{u} =$
$\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} + \left(\frac{\vec{u}\cdot\vec{w}}{\vec{u}\cdot\vec{u}}\right)\vec{u} = Proj_L(\vec{v}) + Proj_L(\vec{w})$.

ii. $Proj_L(k\vec{v}) = kProj_L(\vec{v})$.

Answer: $Proj_L(k\vec{v}) = \left(\frac{\vec{u}\cdot(k\vec{v})}{\vec{u}\cdot\vec{u}}\right)\vec{u} = \left(\frac{k(\vec{u}\cdot\vec{v})}{\vec{u}\cdot\vec{u}}\right)\vec{u} = kProj_L(\vec{v})$.

(b) Let $u = (1, 1)$. Use the above formula for $Proj_L(\vec{x})$ to find the standard matrix P of $Proj_L$.

Answer: Recall that we find the matrix of a linear transformation column by column. Write $P = (\vec{p}_1\vec{p}_2)$. Then

$$p_1 = P\begin{pmatrix} 1 \\ 0 \end{pmatrix} = Proj_L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$p_2 = P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = Proj_L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find the matrix R of the rotation of the plane 90 degrees counterclockwise.

Answer: We get the equality $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ either by using the general formula for the rotation $R = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, with θ equal 90 degrees, or we write $R = (\vec{r_1}\vec{r_2})$ and find R column by column.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation, which first rotates a vector 90 degrees counterclockwise, and then projects the resulting vector onto the line L. Express the standard matrix A of T in terms of the standard matrices P of $Proj_L$ and R of the rotation: $A = \underline{\quad PR \quad}$.
Use this expression to compute A.

Answer: $A = PR = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

(e) (5 **bonus** points) Find a vector $\vec{v}$ in $\mathbb{R}^2$, such that the linear transformation T in part 5d admits the new description $T(\vec{x}) = R(Proj_{\widetilde{L}}(\vec{x}))$, where $\widetilde{L}$ is the line through the origin and $\vec{v}$. Justify your answer.

Answer: Let $\widetilde{P}$ be the standard matrix of $Proj_{\widetilde{L}}$. Then $R\widetilde{P} = PR$. So $\widetilde{P} = R^{-1}PR$. Now it is easy to visualize that the linear transformation, which takes $\vec{x}$ to $R^{-1}PR\vec{x}$, is the projection onto the line $\widetilde{L} = R^{-1}(L)$ obtained from L by rotating 90 degrees clockwise. The line $\widetilde{L}$ in $\mathbb{R}^2$ is cut out by the equation $x_2 = -x_1$. We could choose any non-zero vector $\vec{v}$ in $\widetilde{L}$, for example $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If you have difficulty visualizing, simply compute $R^{-1}PR$ and note that it is indeed the matrix of a projection onto the line spanned by either one of its columns.