Math 235 Midterm 2 Fall 2008

Name:______________________

Justify all your answers. Show all your work!!!

1. (20 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 0 & 2 \\ 2 & 1 & 5 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 1 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: you do **not** have to check that A and B are indeed row equivalent.

a) Determine the rank of A. Explain how it is determined by the matrix B.

b) Find a basis for the kernel $\ker(A)$ of A.

c) Find a basis for the image $\mathrm{im}(A)$ of A.

d) What are all the possible ranks of a 4×4 matrix C, given that C satisfies $CA = 0$ (the product is the zero matrix), where A is the matrix given above? **Justify** your answer! (Additional calculations are not needed).

2. (15 points) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, and W the plane in $\mathbb{R}^3$ spanned by v_1 and v_2.

(a) Find a vector w in W, such that the coordinate vector of w with respect to the basis $\beta := \{v_1, v_2\}$ of W is $[w]_\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(b) Let $v_3 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $u := \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$. Find the coordinate vector of u with respect to the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

3. (22 points) Let $\vec{v} := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Recall that the reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to the line spanned by $\vec{v}$, is given by $T(\vec{x}) = 2\left(\frac{\vec{x} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v} - \vec{x}$.

(a) Show that the matrix of T, with respect to the standard basis, is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show all your work!

(b) Let $\vec{w} := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Compute $T(\vec{v})$ and $T(\vec{w})$.

(c) Use your answer to part 3b in order to find the matrix B of T with respect to the basis $\{\vec{v}, \vec{w}\}$ column by column. Show all your work.

(d) Linear algebra tells us that there exists an invertible matrix S satisfying $AS = SB$. Find S, avoiding any calculations. Explicitly check your answer by multiplying out both sides!

4. (20 points) Set $A := \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be given by $T(M) = MA - AM$.

   (a) Show that the transformation T is linear. In other words, verify the following identities, for any two elements M, N of $\mathbb{R}^{2\times 2}$, and for every scalar k.

      i. $T(M + N) = T(M) + T(N)$.

      ii. $T(kM) = kT(M)$.

   (b) Find a basis for the kernel of T. Show all your work!

5. (23 points) Let P_2 be the vector space of polynomials of degree less than or equal to 2, with real coefficients. Let $T : P_2 \to \mathbb{R}^3$ be the linear transformation given by

$$T(p) := \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}. \qquad (1)$$

   (a) Show that the subset $\{T(1), T(x), T(x^2)\}$ of $\mathbb{R}^3$ is linearly independent.

   (b) Show that $\mathrm{im}(T)$ is the whole of $\mathbb{R}^3$.

   (c) Show that $T : P_2 \to \mathbb{R}^3$ is an isomorphism. Carefully state any criterion you are using!

   (d) Consider next the vector space P_4 of polynomials of degree less than or equal to 4. Let $S : P_4 \to \mathbb{R}^3$ be given by the same formula (1). Find the rank of S.

   (e) Use your answer to part 5d, in order to find a basis for the kernel of S. Explain why the polynomials you found span the kernel, and why they are linearly independent. Hint: Note that $p(x) = x(x - 1)(x + 1)$ belongs to the kernel of S.