Math 235 section 4 Midterm 1 Fall 2008

Name:______________________

1. (15 points) a) Show that the row **reduced** echelon form of the augmented matrix of the system
$$\begin{aligned} x_1 + x_3 - x_4 - 2x_5 &= 2 \\ x_1 + x_2 + 3x_3 &= 1 \\ 2x_1 + 2x_3 + x_4 + 5x_5 &= 1 \end{aligned}$$
is
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -1 \end{pmatrix}.$$
Use at most five elementary operations. Show all your work. Clearly write in words each elementary row operation you used.

b) Find the general solution for the system.
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} =$$

2. (20 points) You are given that the row reduced echelon form of the matrix

$$A = \begin{pmatrix} 3 & 6 & 1 & 2 & 6 & -4 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 1 & 2 & 0 & 0 & 1 & -1 \\ 1 & 2 & 2 & 0 & -1 & 1 \end{pmatrix} \text{ is } B = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

You do **not** need to verify this statement.

(a) Write the general solutions of the system $A\vec{x} = \vec{0}$ in parametric form $\vec{x} = (\text{first free variable})\vec{v}_1 + (\text{second free variable})\vec{v}_2 + \ldots$

(b) Let $T : \mathbb{R}^6 \to \mathbb{R}^4$ be the linear transformations given by $T(\vec{x}) = A\vec{x}$. Find a finite set of vectors in $\ker(T)$, which spans $\ker(T)$. Explain why the set you found spans $\ker(T)$.

(c) Let $\vec{a}_j$ be the j-th column of A. Explain, without any further calculations, why $\vec{a}_6$ belongs to $\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\}$.

(d) Is the image of T equal to the whole of $\mathbb{R}^4$? Justify your answer.

3. (a) (10 points) Determine for which values of k is the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & k+1 \end{pmatrix}$ invertible, and compute the inverse, when it exists.

(b) (2 points) Check that the matrix you found is indeed A^{-1}.

(c) (8 points) Let A, B, C be $n \times n$ matrices, with A invertible, which satisfy the equation $ACA^{-1} - A = B$. Express C in terms of A and B. Show all your work.

4. (20 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with standard matrix A. Assume that the image of T is the whole of $\mathbb{R}^2$. Carfully **justify** your answers to the following questions.

(a) The rank of A is: ____.

(b) Describe geometrically the kernel of T.

(c) Consider the standard matrix A of T. Fix one solution $\vec{p}$ of the equation

$$A\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Show that if a vector $\vec{x}$ is a solution of the above equation, then $\vec{x} - \vec{p}$ belongs to the kernel of T. Show also the converse: If $\vec{x} - \vec{p}$ belongs to the kernel of T, then the vector $\vec{x}$ is a solution of the above equation.

5. (25 points) Let L be the line in $\mathbb{R}^2$ through the origin and the non-zero vector $\vec{u}$. Recall that the projection $Proj_L : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane onto the line L is given by the formula $Proj_L(\vec{x}) = \left(\frac{\vec{u}\cdot\vec{x}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$.

   (a) Use the algebraic properties of the dot product to show that $Proj_L$ is a linear transformation. In other words, verify the following identities, for any two vectors $\vec{v}, \vec{w}$ and for every scalar k.

      i. $Proj_L(\vec{v}+\vec{w}) = Proj_L(\vec{v}) + Proj_L(\vec{w})$.

      ii. $Proj_L(k\vec{v}) = kProj_L(\vec{v})$.

   (b) Let $u = (1, 1)$. Use the above formula for $Proj_L(\vec{x})$ to find the standard matrix P of $Proj_L$.

(c) Find the matrix R of the rotation of the plane 90 degrees counterclockwise.

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation, which first rotates a vector 90 degrees counterclockwise, and then projects the resulting vector onto the line L. Express the standard matrix A of T in terms of the standard matrices P of $Proj_L$ and R of the rotation: $A =$ __________.
Use this expression to compute A.

(e) (5 **bonus** points) Find a vector $\vec{v}$ in $\mathbb{R}^2$, such that the linear transformation T in part 5d admits the new description $T(\vec{x}) = R(Proj_{\widetilde{L}}(\vec{x}))$, where $\widetilde{L}$ is the line through the origin and $\vec{v}$. **Justify** your answer.