MATH 132H SPRING 2004 EXAM 2 SOLUTION

1. (16 points) For each of the following improper integrals, determine if it converges or diverges. If convergent, evaluate the integral. Otherwise, explain why it diverges.

   a) The integral is divergent. Substitute $u = 1 + x^2$ to get:
   $\int_0^\infty \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^\infty \frac{du}{u} = \frac{1}{2}\lim_{t\to\infty}\int_1^t \frac{du}{u} = \frac{1}{2}\lim_{t\to\infty}[\ln(t) - \ln(1)] = \infty.$

   b) Using integration by parts, with $u = \ln(x)$ and $dv = x^{-2}dx$, we get that the integral is convergent.
   $\int_1^\infty \frac{\ln(x)}{x^2}dx = \lim_{t\to\infty}\left\{\left[-\frac{\ln(x)}{x}\right]_1^t - \int_1^t -x^{-2}dx\right\} =$
   $\lim_{t\to\infty}\left\{\left[-\frac{\ln(t)}{t} + \ln(1)\right] - \left[\frac{1}{t} - 1\right]\right\} \overset{L'Hopital}{=} [0+0] - [0-1] = 1.$

2. (a) (12 points) Use the trigonometric substitution $x = \frac{1}{3}\sin(\theta)$, $dx = \frac{1}{3}\cos(\theta)d\theta$:
   $I := \int \sqrt{1-9x^2}dx = \int \sqrt{1 - \frac{9}{9}\sin^2(\theta)}\left[\frac{1}{3}\cos(\theta)\right]d\theta = \frac{1}{3}\int \cos^2(\theta)d\theta =$
   $\frac{1}{3}\int \frac{1+\cos(2\theta)}{2}d\theta = \frac{1}{6}[\theta + \frac{\sin(2\theta)}{2}]d\theta + C =$
   Proceed using the identities $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, $\sin(\theta) = 3x$, and $\cos(\theta) = \sqrt{1-9x^2}$.
   $I = \frac{1}{6}[\arcsin(3x) + 3x\sqrt{1-9x^2}] + C = \frac{\arcsin(3x)}{6} + \frac{x\sqrt{1-9x^2}}{2} + C$

   (b) (3 points) The area enclosed by the ellipse $9x^2 + y^2 = 1$.
   Answer: Area $= 2\int_{-1/3}^{1/3}\sqrt{1-9x^2}dx \overset{\text{part } 2a}{=} \frac{\pi}{3}.$

3. (8 points) $\int \frac{1}{x^2 - x}dx = \int \left[\frac{-1}{x} + \frac{1}{x-1}\right]dx = -\ln|x| + \ln|x-1| + C$

4. (30 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Show all your work! Explain, in particular, which test you used and why the conditions of the test are satisfied.

   a) (8 points) Use the Ratio Test for absolute convergence of $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$.
   $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)(2/3)^{n+1}}{n(2/3)^n} = (2/3)\lim_{n\to\infty}\frac{n+1}{n} = 2/3 < 1.$
   Thus, the series converges absolutely.

   b) (7 points) The series $\sum_{n=1}^{\infty}(-1)^n\frac{n^2-4n}{2n^2+7n+5}$ diverges by the n-term test:
   $\lim_{n\to\infty}\left|(-1)^n\frac{n^2-4n}{2n^2+7n+5}\right| = \frac{1}{2} \neq 0.$

c) (8 points) The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{5n}{n^2+2n}$ is conditionally convergent. We show this in two steps:

Step I: The series of absolute values $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{n^2+2n}$ diverges, by the limit comparison test with the divergent harmonic series:
$\displaystyle\lim_{n\to\infty}\left[\frac{5n}{n^2+2n}\right] / \left[\frac{1}{n}\right] = \lim_{n\to\infty}\frac{5n}{n+2} = 5 > 0.$

Step II: The three conditions of the Alternation Series Theorem are satisfied:
i) The sign is alternating,
ii) $\lim_{n\to\infty} \frac{5n}{n^2+2n} = \lim_{n\to\infty} \frac{5}{n+2} = 0$, and
iii) The sequence $\left\{\frac{5}{n+2}\right\}$ is decreasing.
Thus, the series is conditionally convergent.

d) (7 points) The function $\frac{1}{x[\ln(x)]^2}$ is positive and decreasing for $x \geq 2$. Thus, the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n[\ln(n)]^2}$ converges, by the integral test, since the following improper integral is convergent:

$$\int_2^{\infty}\frac{1}{x[\ln(x)]^2}dx \overset{u=\ln(x)}{=} \int_{u=\ln(2)}^{\infty}\frac{du}{u^2} = \lim_{t\to\infty}\int_{\ln(2)}^{t}\frac{du}{u^2} = \lim_{t\to\infty}\left[-\frac{1}{u}\right]_{\ln(2)}^{t} = \frac{1}{\ln(2)}.$$

5. (15 points) a) A power series representation for the function

$$\frac{x}{1+3x^4} = x\cdot\frac{1}{1-(-3x^4)} = x\cdot\sum_{n=0}^{\infty}(-3x^4)^n = \sum_{n=0}^{\infty}(-3)^n x^{4n+1}.$$

b) The interval of convergence of the power series in part a) is equal to that of the geometric series $\sum_{n=0}^{\infty}(-3x^4)^n$, which is convergent if and only if $|-3x^4| < 1$. Thus, the interval of convergence is $-\left(\frac{1}{3}\right)^{1/4} < x < \left(\frac{1}{3}\right)^{1/4}$.

c) $\displaystyle\int\frac{x}{1+3x^4}dx = \int\left[\sum_{n=0}^{\infty}(-3)^n x^{4n+1}\right]dx = \sum_{n=0}^{\infty}\int(-3)^n x^{4n+1}dx =$
$\displaystyle\sum_{n=0}^{\infty}(-3)^n\frac{x^{4n+2}}{4n+2} + C.$

6. (16 points) a) Use the formula for the coefficient of the Taylor series, in order to determine the Taylor series for $f(x) = \sin(x)$ centered at $a = 0$ (i.e., the Maclaurin series). Show all your work! Credit will **not** be given for an answer without a justification.

The calculation of the Taylor series $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ is carried out in Section 11.10 Example 4 page 765 in the text.

b) Recall Taylor's Remainder Inequality: If $\left|f^{(n+1)}(x)\right| \leq M$, for $|x-a| \leq d$, then the remainder $R_n(x)$, of the Taylor series centered at $x = a$, satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \quad \text{for } |x-a| \leq d.$$

Use Taylor's remainder inequality to determine the **minimal** degree n, of the Taylor polynomial $T_n(x)$ centered at 0, needed to approximate $\sin(0.1)$ to within 5 decimal digits.

Answer: The upper bound M could be chosen as 1, since $\left|\sin^{(n+1)}(x)\right|$ is equal to $|\sin(x)|$ or $|\cos(x)|$ and both are bounded by 1. Taylor's inequality yields

$$\begin{aligned} |\sin(x) - T_n(x)| = |R_n(x)| &< \frac{|x|^{n+1}}{(n+1)!} \\ |\sin(0.1) - T_n(0.1)| = |R_n(0.1)| &< \frac{(0.1)^{n+1}}{(n+1)!} \end{aligned} \tag{1}$$

In order to assure, that the error $|R_n(0.1)|$ is less than 0.00001, we need to choose n, so that the right hand side of the inequality (1) is less than $0.00001 = (10)^{-5}$. For $n = 1$ the right hand side of equation (1) is $\frac{1}{200}$, for $n = 2$ it is $\frac{1}{6000}$, and for $n = 3$ it is $\frac{1}{240000}$. Thus, $n = 3$.

Note, that $T_3(x) = x - \frac{x^3}{3!}$, $T_3(0.1) = 0.1 - \frac{1}{6000}$, and

$$|\sin(0.1) - T_3(0.1)| = |R_3(0.1)| < \frac{1}{240000}.$$