Math 132H–SPRING 2006 Solution to EXAM 1

1. (8) Rewrite $\frac{1+\sqrt{x}}{3x} = \frac{1}{3}(x^{-1} + x^{-1/2})$ to obtain:
$\int 5e^{2x} - \frac{1+\sqrt{x}}{3x} + \frac{3}{1+x^2} dx = \frac{5}{2}e^{2x} - \frac{1}{3}\ln|x| - \frac{2}{3}x^{1/2} + 3\arctan(x) + C.$

2. (8) Use the substitution $u = \cos(x)$ and the identity $\sin^4(x) = (1-\cos^2(x))^2$ to obtain: $\int \sin^5(x)\cos^4(x)dx = \frac{(\cos(x))^9}{9} - \frac{2}{7}(\cos(x))^7 + \frac{(\cos(x))^5}{5} + C$

3. (8) Use substitution $u = x^2$ to obtain: $\int x^3\sin(x^2)dx = \frac{1}{2}\int u\sin(u)du.$

   Next, use integration by parts to obtain: $\frac{1}{2}\int u\sin(u)du = \frac{1}{2}\left\{\sin(x^2) - x^2\cos(x^2)\right\} + C.$

4 (15)

a) $\frac{\partial}{\partial x}\int_1^{x^2} \sin(t+t^2)dt = \sin(x^2+x^4)\cdot 2x,$
by the Chain Rule (with $F(u) = \int_1^u \sin(t+t^2)dt$ and $u(x) = x^2$).

b) $\frac{\partial}{\partial x}\int_x^{10} \ln(1+t^2)dt = -\frac{\partial}{\partial x}\int_{10}^{x} \ln(1+t^2)dt = -\ln(1+x^2).$

c) The definite integral $\int_0^\pi \sin(x)dx$ is constant, so its derivative vanishes.

5 (15) a) Sketch the region to the right of the y-axis bounded by the y-axis, the curve $y = \frac{12}{2+x^2}$, and the curve $y = x^2 - 2$.

The two curves meet at $(2,2)$.

b) Use the Cylindral-shells method to find the volume of the solid obtained by revolving about the y-axis the region in part a).

Answer: The radius is x, the height of the cylindral shell is $\frac{12}{2+x^2} - (x^2-2)$, so the volume is given by:

$$Vol = \int_0^2 2\pi x\left[\frac{12}{2+x^2} - (x^2-2)\right]dx = 2\pi\left[6\ln(2+x^2) - \frac{x^4}{4} + x^2\right]_0^2 = 12\pi\ln(3).$$

6 a) (9) Approximate the integral $\int_0^1 e^{(x^2)}dx$ by a Riemann sum that uses 5 equal-length sub-intervals and **left**-hand endpoints as sample points. (Show the individual terms of the Riemann sum before you calculate the value of the sum).

Answer: $0.2\left[e^0 + e^{(.2)^2} + e^{(.4)^2} + e^{(.6)^2} + e^{(.8)^2}\right] \approx 1.3088$

b) (6) Interpret the limit $\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}\cdot\left(\frac{1}{n}\right)$ as the area of a region. Justify this interpretation! Graph this region. Use your interpretation to evaluate the limit.

Answer: The equality $\sum_{i=1}^{n}\sqrt{1-\left(\frac{i}{n}\right)^2}\cdot\left(\frac{1}{n}\right) = \sum_{i=1}^{n} f(x_i^*)\Delta x$ holds, if we take $f(x) = \sqrt{1-x^2}$, sample points $x_i^* = \frac{i}{n}$, and $\Delta x = \frac{1}{n}$. The equality $\Delta x = \frac{1}{n}$ means that the endpoints a, b of the interval satisfy $b - a = n\Delta x = 1$. The sample points would be the right end-points $x_i = a + i \cdot \frac{\Delta x}{n}$, if $a = 0$. Thus, the Riemann sums approximate and converge to $\int_0^1 \sqrt{1-x^2}dx$. The integral is represented geometrically by the area of one quarter of the unit circle. It is hence equal to $\frac{\pi}{4}$.

7 The graph of the curve $x = (y^3 + 5y^2 + y)^{\frac{1}{3}}$, $y > 0$, is revolved about the y-axis to form the outer surface of a water container. Water is being poured in at a constant rate of 10 centimeters cubed per second. Assume the x and y units are in centimeters.

a) (9) Use the disk (washer) method to set-up an integral for the volume $V(h)$ of the water in the container, when the water level reaches the horizontal line $y = h_{\text{cm}}$ (for some constant height h). Do **NOT** evaluate the integral.

Answer: $V(h) = \int_0^h \pi(y^3 + 5y^2 + y)^{2/3} dy$

b) (6) Use the fact that $\frac{\partial V(h(t))}{\partial t} = 10_{\text{cm}^3/\text{sec}}$, the Fundamental Theorem of Calculus, and the Chain Rule, to show that the rate of change $\frac{\partial h}{\partial t}$, of the height $h(t)$ of the water in the container with respect to time, is equal to 10 divided by the horizontal-surface-area of the water

$$\frac{\partial h}{\partial t} = \frac{10}{\pi(h^3 + 5h^2 + h)^{2/3}} \text{ cm/sec.} \qquad (1)$$

Answer: $10 = \frac{\partial V(h(t))}{\partial t} \overset{\text{Chain Rule}}{=} \frac{\partial V(h)}{\partial h} \cdot \frac{\partial h}{\partial t}$.
The Fundamental Theorem of Calculus and part a) yield:
$\frac{\partial V(h)}{\partial h} = \pi(h^3 + 5h^2 + h)^{2/3}$.
Substitute the latter equation in the former and solve for $\frac{\partial h}{\partial t}$ to get equation (1).

8 (16) A ball is thrown upward from a tower window, 200 feet above the ground, with initial velocity $v_0 = 48$ feet per second. Its acceleration, t seconds afterwards, is $v'(t) = -32$ ft/sec^2.

a) The velocity $v(t)$ of the ball t seconds after it is thrown, but before it hits the ground, is $v(t) = -32t + C$, and the constant C is 48, since $v(0) = 48$.

b) The height $h(t)$ of the ball above the ground t seconds after it is thrown is

$$h(t) = 200 + \int_0^t v(t)dt = 200 + \int_0^t 48 - 32t dt = 200 + 48t - 16t^2.$$

c) The total distance traveled by the ball during the time interval $0 \le t \le 2$ seconds is $\int_0^2 |v(t)|dt$. The velocity is positive for $0 < t < \frac{3}{2}$ and negative for $\frac{3}{2} < t < 2$. Thus, the total distance traveled is

$$\int_0^{3/2} 48 - 32t dt + \int_{3/2}^2 -48 + 32t dt = 36 + 4 = 40.$$