1. (14 points) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Show all your work! Explain, in particular, which test you used and why the conditions of the test are satisfied.

7 points a) $\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{\sqrt{n-1}}{2n^2+\sqrt{n}}}_{a_n}$ $|a_n| < \frac{\sqrt{n}}{2n^2} < \underbrace{\frac{1}{n^{3/2}}}_{b_n}$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, by the p-test.
Absolutely convergent, by the comparison test.

7 points b) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

$\int_2^{\infty} \frac{1}{x \ln(x)} dx =$ $u = \ln(x)$, $du = \frac{1}{x} dx$ $\int_{\ln(2)}^{\infty} \frac{1}{u} du = \lim_{t \to \infty} \ln(t) - \ln(\ln(2)) = +\infty$

Divergent.

2. (18 points) Compute the following integrals **algebraically**. Show all your work!

6 pts a) $\int_0^{\pi/2} \underbrace{\sin^3(\theta)}_{\sin(\theta)\cdot[1-\cos^2(\theta)]} \cos^2(\theta)d\theta = \int_0^{\frac{\pi}{2}} [\cos^2(\theta) - \cos^4(\theta)]\,(-d\cos(\theta)) =$ $\boxed{u = \cos(\theta)}$

$\cos(\frac{\pi}{2}) = 0$

$= -\int_{1=\cos(\theta)} u^2 - u^4\, du = \int_0^1 u^2 - u^4\, du = \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15} = 1{,}3\dot{3}$

6 pts b) $\int \underset{u}{x} \sin(2x)dx = x\cdot\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,dx =$

$$\boxed{\begin{array}{ll} u & dv \\ du = dx & v = -\frac{1}{2}\cos(2x) \end{array}}$$

$-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

6 pts

c) $I = \int \frac{x^2+4x+1}{x^2+4}dx = \int 1 + \frac{4x-3}{x^2+4}\,dx = \underbrace{\int 1\,dx}_{=x} + 2\underbrace{\int \frac{2x}{x^2+4}\,dx}_{=\ln(x^2+4)} - 3\int \frac{dx}{x^2+4}$

$$\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$$

$$I = x + 2\ln(x^2+4) - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C$$

3. (10 points) Find the volume of the infinite solid of revolution obtained by rotating the curve $y = \frac{1}{x}$ around the x-axis, over the interval $[1, \infty)$.

$$Vol = \int_1^{\infty} \pi \left(\frac{1}{x}\right)^2 dx = \lim_{t\to\infty} \int_1^t =$$

$$= \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \pi \lim_{t\to\infty}\left(-\frac{1}{t} - (-1)\right) = \pi$$

(8 pts)

4. (14 points) a) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \underbrace{\frac{x^{2n+1}}{n!(2n+1)}}_{a_n}$. Show all your work!

Ratio-Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{\overbrace{2(n+1)+1}^{2n+3}}}{(n+1)!\,(2(n+1)+1)} \cdot \frac{n!\,(2n+1)}{|x|^{2n+1}} = \underbrace{\frac{|x|^2}{(n+1)}}_{\downarrow\, 0} \cdot \underbrace{\frac{2n+1}{2n+3}}_{\downarrow\, n\to\infty\; 1} = 0$$

Interval: $(-\infty, \infty)$

(6 pts)

b) Denote by $f(x)$ the sum of the power series in part 4a. Show that the derivative $f'(x)$ is equal to $e^{(x^2)}$.

$$f'(x) = \sum_{n=0}^{\infty} \frac{(2n+1)\,x^{2n}}{n!\,(2n+1)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$$

Now $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ Take $u = x^2$

6 pts

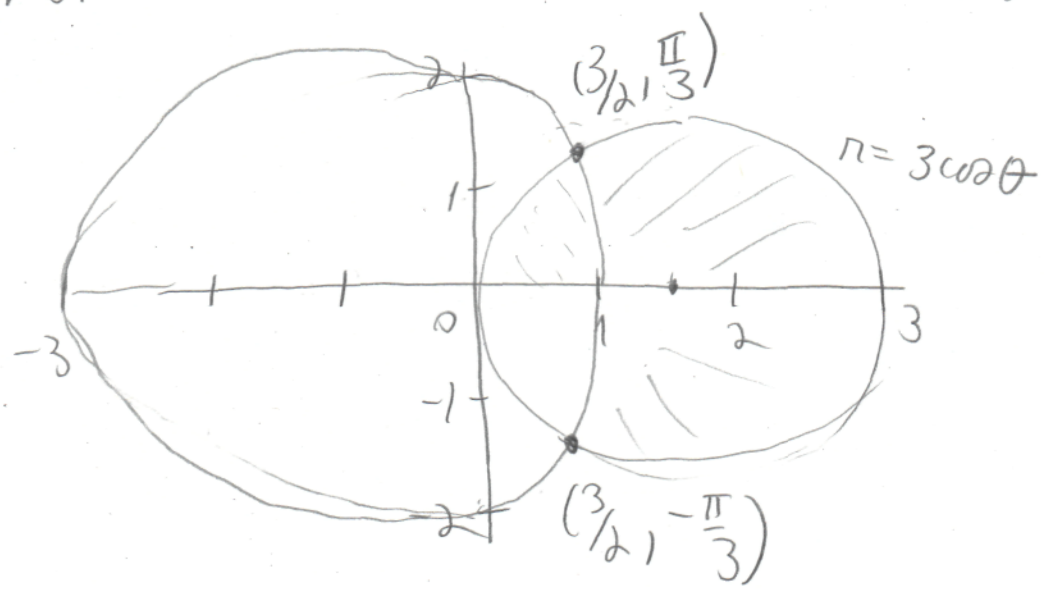
5. (14 points) a) Graph the two polar curves $r = 3\cos(\theta)$ and $r = 2 - \cos(\theta)$ on the same plane. Find polar coordinates for each point of intersection.

θ	$2-\cos(\theta)$
0	1
$\frac{\pi}{3}$	$2-\frac{1}{2}=3/2$
$\frac{\pi}{6}$	$2-\frac{\sqrt{3}}{2} \simeq 1,134$
$\frac{\pi}{2}$	2
π	3

$r^2 = 3r\cos(\theta)$

$x^2+y^2 = 3x$

$(x-\frac{3}{2})^2+y^2 = (\frac{3}{2})^2$ circle of radius $\frac{3}{2}$ centred at $(\frac{3}{2},0)$



Points of intersection

$$3\cos(\theta) = 2-\cos(\theta)$$

$$4\cos(\theta) = 2$$

$$\cos(\theta) = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3} + 2n\pi$$

$$(r,\theta) = \left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, -\frac{\pi}{3}\right)$$

5 pts

b) Set up a definite integral representing the area of the region that lies *inside* $r = 3\cos(\theta)$ and *outside* $r = 2 - \cos(\theta)$.

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left[r_1(\theta)^2 - r_2(\theta)^2\right] d\theta = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[(3\cos(\theta))^2 - \underbrace{(2-\cos(\theta))^2}_{\cos^2(\theta) - 4\cos(\theta) + 4}\right] d\theta$$

3 pts

c) Evaluate the integral in part 5b algebraically, showing all your work.

$$I = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[(9-1)\underbrace{\cos^2(\theta)}_{= \frac{1+\cos(2\theta)}{2}} + 4\cos(\theta) - \cancel{4}\right] d\theta = \frac{4}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\cos(\theta) + \cos(2\theta)\right] d\theta$$

$\cancel{4} + 4\cos(2\theta)$

$$= 2\left[\sin(\theta) + \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = 2\left[\left(\frac{\sqrt{3}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)}{2}\right) - \left(-\frac{\sqrt{3}}{2} - \frac{\left(\frac{\sqrt{3}}{2}\right)}{2}\right)\right]$$

$$= 3\sqrt{3}$$

6. (14 points) A basketball is thrown from the ground. Its position, t seconds after it was thrown, is given by the parametrized curve

5 pts

$$x(t) = 40t \quad \text{and} \quad y(t) = 40t - 10t^2.$$

a) Find the equation of the tangent line at $t = 0$ and use it to determine the angle at which the basketball is thrown?

$$\left.\frac{\partial y}{\partial x}\right|_{t=0} = \frac{\partial y/\partial t}{\partial x/\partial t} = \frac{40-20t}{40} = \left.1-\frac{t}{2}\right|_{t=0} = 1$$

angle $45^\circ = \frac{\pi}{4}$

(3 pts) (1 pt) (1 pt)

3 pts

b) What is the maximal height reached by the basketball?

$$0 = \frac{\partial y}{\partial t} = 40 - 20t \qquad t = 2.$$

$$y(2) = 80 - 40 = 40.$$

6 pt

c) Express as a definite integral the distance traveled by the basketball until it hits the ground. Do NOT evaluate the integral. Hint: Consider the length of the parametrized curve.

Time it hits the ground: $0 = y(t) = 40t - 10t^2 = t(40-10t)$

(2 pt)

$t = 0$, $\boxed{t = 4}$

$$\int_0^4 \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\, dt = \int_0^4 \sqrt{40^2 + (40-20t)^2}\, dt$$

6 pts

7. (16 points) a) Use the formula for the coefficient of the Taylor series, in order to determine the Taylor series for $f(x) = \ln(x)$ centered at $a = 1$. Show all your work! Credit will **not** be given for an answer without a justification.

n	$f^{(n)}(x)$	$f^{(n)}(1)/n!$
0	$\ln(x)$	0
1	$\frac{1}{x}$	1
2	$(-1)x^{-2}$	$-1/2$
3	$2x^{-3}$	$2/3! = \frac{1}{3}$
4	$-2\cdot 3x^{-4}$	$-\frac{1}{4}$
n	$(n-1)!(-1)^{n-1}x^{-n}$	$\frac{(-1)^{n-1}}{n}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{(x-1)^3}{3} + \cdots (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(x-1)^n}{n}$$

4 pts

b) Use the identity $\frac{1}{x} = \frac{1}{1+(x-1)}$ and the formula for the sum of a geometric series, in order to find the Taylor series for $f(x) = \dfrac{1}{x}$ centered at $a = 1$.

$$\frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty}(-1)^n (x-1)^n$$

3 pts

c) Explain how your answers to parts 7a) and 7b) should be related and check this relation.

Differentiation term by term

yields: $7b = \frac{\partial}{\partial x} 7a$

5 pts

d) Recall Taylor's Remainder Inequality: If $\left|f^{(n+1)}(x)\right| \leq M$, for $a \leq x \leq a+d$, then the remainder $R_n(x)$, of the Taylor series centered at $x = a$, satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \quad \text{for } a \leq x \leq a+d$$

(and similarly for the interval $a-d \leq x \leq a$). Use Taylor's remainder inequality to determine the **minimal** degree n, of the Taylor polynomial $T_n(x)$ centered at $a = 1$, needed to approximate $\ln(1.4)$ to within 3 decimal digits. Justify your answer!

$$\left|f^{(n)}(x)\right| = \left|(n-1)!\,(-1)^{n-1}x^{-n}\right| = (n-1)!\,|x|^{-n} < (n-1)! \quad \text{on } (1, 1.4)$$

$$\left|f^{(n+1)}(x)\right| \leq n! = M \quad \text{on } (1, 1.4)$$

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-1|^{n+1} = \frac{|x-1|^{n+1}}{(n+1)}$$

$$|R_n(1.4)| \leq \frac{(0.4)^{n+1}}{n+1}$$

n	$\frac{(0.4)^{n+1}}{n+1}$
1	0.08
2	0.0213
3	0.0064
4	0.002
5	$6.8 \cdot 10^{-4}$

$\boxed{n = 5}$