Disclaimer: This review sheet serves to give a highlight of the topics covered after Midterm #2. It does NOT replace your textbook and/or your lecture notes.

Comments about the practice exams/homework:
- practice exams are on the course website — these are taken verbatim from old exams and may NOT cover the same materials as we do
- the practice exams are intended to give you an IDEA what the questions are like; your homework problems are indented to give you a chance to LEARN the course materials. The actual exam MAY contain problems DIFFERENT from those in the practice exams and/or homeworks!
- for additional practice: try the end-of-chapter review problems

Other comments about your exams:
- SHOW YOUR WORK!
- study the examples in your textbook
- the final exam is TWO HOURS LONG

11.1, 11.2

- a series $\sum a_n$ converges if and only if the sequence of partial sums $s_n = a_1 + \cdots + a_n$ converges
- basic series:

<table>
<thead>
<tr>
<th>shape</th>
<th>harmonic series</th>
<th>geometric series</th>
<th>$p$-series</th>
<th>telescopic series</th>
</tr>
</thead>
<tbody>
<tr>
<td>behavior</td>
<td>always divergent</td>
<td>conv if $</td>
<td>r</td>
<td>&lt; 1$</td>
</tr>
<tr>
<td></td>
<td>$\sum_{n=1}^{\infty} \frac{1}{n}$</td>
<td>$\sum_{n=0}^{\infty} ar^n$</td>
<td>$\sum_{n=1}^{\infty} \frac{1}{n^p}$</td>
<td>div if $p \leq 1$</td>
</tr>
</tbody>
</table>

- $n$-term test:
  - if $\lim_{n\to\infty} a_n \neq 0$ or if this limit does not exist, then $\sum a_n$ diverges
  - warning: the $n$-term test is only a ONE-WAY test – if $\lim_{n\to\infty} a_n = 0$ then the series MAY or MAY NOT converge !
- note the difference between a $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and a geometric series $\sum_{n=1}^{\infty} \frac{a}{n^p}$ — note the different locations where $n$ appear

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Flow-chart for convergent test:

Does $\sum_n a_n$ converge?

- $\lim_{n \to \infty} a_n = 0$?
  - no → divergent

  - no → all terms positive?
    - no → AST? ratio test?
    - yes → integral test? comparison tests? ratio test?
  - yes → proceed accordingly

11.3, 11.4

- integral test
  - suppose $a_n = f(n)$ for some continuous, decreasing, positive function $f$, then
    \[ \int_1^\infty f(x) \, dx \text{ converges } \iff \sum a_n \text{ converges} \]
  - error estimate for integral test:
    \[ \int_{n+1}^{\infty} f(x) \, dx < s_n - s_n < \int_n^\infty f(x) \, dx \]

- comparison test
  - suppose both $\sum a_n, \sum b_n$ both have positive terms.
    * if $a_n \leq b_n$ for all large $n$ and if $\sum b_n$ converges, then $\sum a_n$ converges
    * if $a_n \geq b_n$ for all large $n$ and if $\sum a_n$ diverges, then $\sum b_n$ diverges
  - error estimate for the comparison test: if $0 \leq a_n \leq b_n$ for all $n$ and if $\sum b_n$ converges, then
    \[ 0 \leq s - s_n < b_{n+1} + b_{n+2} + \ldots \]

- limit comparison test
  - suppose both $\sum a_n, \sum b_n$ both have positive terms. If $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and is finite and non-zero, then
    \[ \sum a_n \text{ converges } \iff \sum b_n \text{ converges} \]
  - NOTE: there is NO error estimate for the limit comparison test!

- to use the two comparisons we need something to compare! Good candidates: $p$-series and geometric series. Also, remember the relative size of functions:
  \[
  \text{powers of log } n \ll \text{positive powers of } n \ll (\text{fixed number } > 1)^n \ll n! \ll n^n
  \]
11.5:

- **alternating series test**
  - If
    - every \(b_i > 0\)
    - \(\lim_{n \to \infty} b_n = 0\)
    - \(b_1 \geq b_2 \geq b_3 \geq \cdots\)
  
  then the alternating series \(\sum_{n=1}^{\infty} (-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 \pm \cdots\) converges

- **error estimates**: suppose the alternating series \(b_1 - b_2 + b_3 - b_4 \pm \cdots\) satisfies the AST, then \(|s - s_n| \leq b_{n+1}\)

11.6:

- \(\sum a_n\) is called **absolutely convergent** if \(\sum |a_n|\) converges
- \(\sum a_n\) is called **conditionally convergent** if \(\sum a_n\) converges but \(\sum |a_n|\) does not.
  - Equivalently: \(\sum a_n\) converges but not absolutely converges
- **Theorem**: absolutely convergent \(\implies\) convergent
  - converse is FALSE! E.g. \(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\) converges but NOT absolutely converges.
- **Ratio Test**:
  
  \[
  \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \left| \begin{array}{c}
  \text{exists and is } < 1 \\
  \text{exists and is } > 1; \text{ or if the limit is } +\infty \\
  \text{exists and is } = 1; \text{ or if the limit does not exist}
  \end{array} \right|
  \]
  
  **Conclusion**: absolute conv. divergence inconclusive

11.8, 11.9

- given a power series, use the ratio test to find its radius of convergence, interval of convergence and its center
  
  - don’t forget to check the end points when you try to determine the IOC!
- can get power series representation of functions by manipulating geometric series – the basic idea is to turn your expression into something that resemble \(\frac{1}{1 - \text{blah}}\):
  
  - Step 0: make sure the numerator is 1
  - Step 1: make sure the denominator begins with the coefficient 1
  - Step 2: arrange the denominator so that it looks like \(1 - \text{blah}\)
  - Step 3: apply the geometric series formula – so long as \(|\text{blah}| < 1\)
- within the interval of convergence you can differentiate and/or integrate a power series term-by-term
  
  - use this to get power series representation for \(\arctan x\) and for \(\ln(1 - x)\) (both for \(|x| < 1\))
Taylor polynomials

- know how to compute Taylor series & MacLaurin series
  - Taylor series with center $a$: \[ \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \]
  - MacLaurin series: Taylor series with center 0; i.e. \[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \]

- know the series (and the interval of convergence) of the basic functions:
  \[ e^x, \sin x, \cos x, \arctan x, \ln(1 - x), \frac{1}{1 - x} \]

- applications: e.g. computing indefinite integrals; computing limits; estimating definite integrals (in conjunction with error estimates)