Solutions to the review problems

1.
First we find the points where the two curves coincide: $x^3 - x = 3x$ gives 0 and ± 2 as solutions. Since both functions are odd, the area is given by two times the difference of the areas bounded by each of the two curves and $x = 0$ and $x = 2$ (but we stuff them in one integral immediately)

$$\begin{aligned} 2\int_0^2 |3x - (x^3 - x)|\, dx &= 2\int_0^2 4x - x^3\, dx \\ &= 2(2x^2 - \tfrac{1}{4}x^4|_0^2) \\ &= 16 - 8 - (0 - 0) = 8 \end{aligned}$$

2.
First we find the general antiderivative with a "mental" substitution $u = 2x$:

$$f(x) = \int e^{2x} = \frac{1}{2}e^{2x} + C$$

then we have to determine the constant C using the condition $f(0) = 5$:

$$f(0) = \frac{1}{2} + C = 5 \Rightarrow C = 4.5$$

Finally we can find $f(10) = \frac{1}{2}e^{20} + 4.5 = \frac{1}{2}(e^{20} + 9)$.

3.
a) nothing to do...
b) We know that $f(5) - f(-5) = \int_{-5}^{5} f'(x)\, dx$ but since $f'(x) \geq 0$, the right hand side is just the area bounded by f', the y-axis and $x = -5$ and $x = 5$. Thus $f(5) - f(-5) = 29$. Since $f(-5)$ was given to be 10 we find

$$f(5) = 29 + 10 = 39.$$

c) Since $f'(x) \geq 0$ for $-5 \leq x \leq 5$ we know that $\int_{-5}^{5} |f'(x)|\, dx = \int_{-5}^{5} f'(x)\, dx = 29$.

4.
a) We use the ratio test: $a_n = \frac{4x^{2n}}{3^n n!}$ so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{4x^{2(n+1)}}{3^{(n+1)}(n+1)!}\frac{3^n n!}{4x^{2n}}\right| = \lim_{n\to\infty} \frac{x^2}{3(n+1)} = 0$$

(remember that $(k+1)! = (k+1)k!$). So the series converges for all x and the radius of convergence is ∞.
b) The way it is stated the series is no power series, so we can not compute a radius of convergence. If we assume $\sum_{n=0}^{\infty} \frac{4^n x^n}{3^n}$, we can use the root test: Now $a_n = \frac{4^n x^n}{3^n}$ and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{4|x|}{3}\right)^n} = \lim_{n\to\infty} \frac{4|x|}{3} = \frac{4}{3}|x|$$

For the series to converge we need this limit to be smaller than 1: $\frac{4}{3}|x| < 1$. This leaves us with $|x| < \frac{3}{4}$. Therefore the radius of convergence is $\frac{3}{4}$.

5.
a) Because of the factorial in the denominator, we try the ratio test: We have $a_n = \frac{4n+1}{(2n+1)!}$ so

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{4(n+1)+1}{(2(n+1)+1)!}\frac{(2n+1)!}{4n+1}\right| \\
&= \lim_{n\to\infty}\frac{4n+5}{(2n+1+2)!}\frac{(2n+1)!}{4n+1} \\
&= \lim_{n\to\infty}\frac{4n+5}{(2n+1+2)(2n+1+1)}\frac{1}{4n+1} \\
&= \lim_{n\to\infty}\frac{4n+5}{(2n+3)(2n+2)(4n+1)}\frac{\frac{1}{n^3}}{\frac{1}{n^3}} \\
&= \lim_{n\to\infty}\frac{\frac{4}{n^2}+\frac{5}{n^3}}{(2+\frac{3}{n})(2+\frac{2}{n})(4+\frac{1}{n})} \\
&= 0
\end{aligned}$$

by the standard argument for rational functions. Since this limit is smaller than 1 the series is absolutely convergent.
b) This series is in fact the power series in problem 4 b) evaluated at 1 Since there the radius of convergence was $\frac{3}{4}$ which is smaller than 1 we know that the series is divergent and therefore not absolutely convergent. (An other way to argue would be to state that the series is in fact a geometric series with $r = \frac{4}{3} > 1$.)
c) We can use the comparison test. The series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent. In our case we have $a_n = \frac{1}{4^n-2^n}$ and since $4^n > 2^n$ for $n > 0$ all a_n are positive. So we know $|a_n| = a_n$. Now we can estimate our series with a geometric series from above: Certainly $\frac{1}{4^n-2^n} = \frac{1}{2^n(2^n-1)} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ is convergent (it is a geometric series with $r = 1/2$) our series must be convergent too. Again: since all terms are positive anyways the series is absolutely convergent then.

6.
By definition the improper integral is computed in the following way:

$$\begin{aligned}
\int_5^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_5^t \frac{1}{x}\,dx = \lim_{t\to\infty}\ln(x)|_5^t \\
&= \lim_{t\to\infty}\ln(t) - \ln(5) = \infty
\end{aligned}$$

Thus the integral is divergent.

7.
We can use the ratio test once again: $a_n = \frac{nx^n}{3^n}$ so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{(n+1)}}{3^{(n+1)}}\frac{3^n}{nx^n}\right| = \lim_{n\to\infty}\frac{(n+1)|x|}{3n} = \frac{|x|}{3}$$

For the series to be convergent for a particular x this limit has to be smaller than 1, so the series converges for $|x| < 3$ for sure (we don't know what happens at ± 3) and the radius of convergence is 3.

8. a) This is sort of "reverse engineering": We know, that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

holds, so a geometric series with $a = 1$ and $r = -x^2$ would have the right sum (and it would converge for $|x| < 1$). We only have to bring it to a power series form. The terms of the geometric series read $a_n = (-x^2)^{n-1} = (-1)^{n-1}x^{2n-2}$ The power series is therefore as follows:

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}, \quad |x| < 1.$$

b) We can integrate power series as we would integrate polynomials. Using the hint and part a) we find that

$$\tan^{-1} + C = \int \frac{1}{1+x^2}\, dx = \int \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\, dx$$

for $|x| < 1$. Now

$$F(x) = \int \sum_{n=1}^{\infty}(-1)^{n-1}x^{2n-2}\, dx = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}x^{2n-1} + C$$

We are left with finding the correct value of C: $\tan^{-1}(0) = 0$ and $F(0) = C$ so C must be zero. Thus we have found

$$\tan^{-1}(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}x^{2n-1}, \quad |x| < 1.$$

9.
The fundamental theorem of calculus tells us that if $G(x) = \int_a^x g(t)\, dt$ then $G'(x) = g(x)$ holds. However, here the upper bound of the integral does not read x but $3x$. If we define $G(x) = \int_3^x e^{t^2}\, dt$ then $F(x) = G(3x)$. Therefore we have

a) by chain rule $F'(x) = G'(3x)3 = 3e^{(3x)^2}$.
b) by a) $F'(0) = 3e^{(3\cdot 0)^2} = 3$.
c) by the laws for definite integrals $F(1) = \int_3^3 e^{t^2}\, dt = 0$.

An other way for problem 9 would be to use substitution rule to transform $\int_3^{3x} e^{t^2}\, dt = \int_1^x 3e^{(3t)^2}\, dt$ first and then use the FTC directly.

10.
a) The formula for the derivative of a parametric curve gives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t + 2t}{\cos(t) + 2}.$$

b) we have not covered arc length
c) we still have not covered arc length
d) the equation reads in general

$$y - y_0 = \frac{dy}{dx}(0)(x - x_0)$$

In our case this gives

$$y + 3 = \frac{1}{3}(x - 1).$$

11.
The divergence test tells us that a necessary condition for a series to converge is, that the limit of the terms a_n is 0 as n goes to infinity. But here we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^3}{2n^3} = \lim_{n\to\infty} 2 = 2 \neq 0$$

So the series must be divergent.

12.
We start with the observation, that $e^{2x} = (e^x)^2$. Now we can substitute $u = e^x$ which gives $du = e^x dx$. Thus (remembering the hint for 8 b))

$$\int \frac{e^x}{e^{2x} + 1}\, dx = \int \frac{1}{u^2 + 1}\, du = \tan^{-1}(u) + C = \tan^{-1}(e^x) + C.$$