Section 11.1 problem 98:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.

(a) Assume that $\lim_{n\to\infty} a_{2n} = L$ (*) and $\lim_{n\to\infty} a_{2n+1} = L$ (**).

(*) is equivalent to the statement:
For every $\varepsilon > 0$ the inequality $|L - a_{2n}| < \varepsilon$ holds for all but finitely many n.

(**) is equivalent to the analogous statement with the inequality $|L - a_{2n+1}| < \varepsilon$.

Let $\varepsilon > 0$ and let S be the following subset of positive integers
$S = \{n : |L - a_n| \geq \varepsilon\}$.
The set of even integers in S is finite, by (*), and the set of odd integers in S is finite, by (**). Hence S is finite. Thus, the inequality $|L - a_n| < \varepsilon$ holds for all but finitely many integers n. So $\lim_{n\to\infty} a_n = L$, by definition of the limit of a sequence.

$$\text{Set } f(x) = \frac{4+3x}{3+2x}$$

Part (b): (i) Let $b_n = a_{2n}$, $c_n = a_{2n+1}$.

We need to show that $b_{n+1} = f(b_n)$ and $c_{n+1} = f(c_n)$. In other words, we need to show that

$$\boxed{a_{n+2} = f(a_n).}$$

$$a_{n+2} \underset{\substack{\uparrow \\ \text{def of } \{a_n\}}}{=} 1 + \frac{1}{1+a_{n+1}} \underset{\substack{\uparrow \\ \text{def of } \{a_n\}}}{=} 1 + \left[\frac{1}{1+\underbrace{\left(1+\frac{1}{1+a_n}\right)}_{2+\frac{1}{1+a_n} = \frac{3+2a_n}{1+a_n}}}\right] =$$

$$= 1 + \left[\frac{1+a_n}{3+2a_n}\right] = \frac{4+3a_n}{3+2a_n} = f(a_n).$$

b(ii) f is increasing on $(0,\infty)$ and $f(\sqrt{2}) = \sqrt{2}$.

$$f(\sqrt{2}) = \frac{4+3\sqrt{2}}{3+2\sqrt{2}} = \frac{4+3\sqrt{2}}{3+2\sqrt{2}} \cdot \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{12+(9-8)\sqrt{2}-12}{9-8}$$

$$= \sqrt{2}.$$

$$f'(x) = \frac{3(3+2x) - (4+3x)\cdot 2}{(3+2x)^2} = \frac{1}{(3+2x)^2} \underset{\boxed{\text{for } x>0}}{>} 0.$$

Hence, f is (strictly) increasing for $x > 0$. I.e., if $x_1 < x_2$, then $f(x_1) < f(x_2)$.

b(iii) $\{c_n = a_{2n+1}\}_{n=1}^{\infty}$ is increasing and bounded above by $\sqrt{2}$.

We will show first two properties of f:

(I) $f(x) - x > 0$, for $x \in (0,\sqrt{2})$.

(II) $f(x) < \sqrt{2}$, for $x \in (0,\sqrt{2})$.

Proof of (I): Let $g(x) = f(x) - x$. Then for $x > 0$

$$g'(x) = f'(x) - 1 = \frac{1}{(3+2x)^2} - 1 < \frac{1}{9} - 1 = -\frac{8}{9} < 0.$$

Hence, g is DECREASING (STRICTLY) on $(0, \infty)$. Note that $g(\sqrt{2}) = f(\sqrt{2}) - \sqrt{2} = 0$.
Hence, for $0 < x < \sqrt{2}$, we have:
$g(x) > g(\sqrt{2}) = 0$. So $f(x) - x > 0$, for $x \in (0, \sqrt{2})$.

Proof of II: If $0 < x < \sqrt{2}$, then
$f(x) < f(\sqrt{2}) = \sqrt{2}$, since f is increasing on $(0, \infty)$.

We are ready to prove that $\{c_n = a_{2n+1}\}_{n=0}^{\infty}$ is increasing and bounded above by $\sqrt{2}$.
Check for $c_0 = a_1 = 1 < \sqrt{2}$. Assume $0 < c_n < \sqrt{2}$.
Then $c_{n+1} = f(c_n) \underset{\text{by (I), since } 0 < c_n < \sqrt{2}}{>} c_n$, so $\{c_n\}$ is increasing

and $c_{n+1} = f(c_n) \underset{\text{by (II), since } 0 < c_n < \sqrt{2}}{<} \sqrt{2}$.

We see that if the finite sequence $\{c_0, \ldots, c_n\}$ is increasing and bounded above by $\sqrt{2}$, then so is $\{c_0, \ldots, c_n, c_{n+1}\}$.
Thus, the infinite sequence $\{c_n\}_{n=0}^{\infty}$ is increasing and bounded by $\sqrt{2}$ by the Principle of Mathematical Induction.

$\{b_n = a_{2n}\}_{n=1}^{\infty}$ is decreasing and bounded below by $\sqrt{2}$.

b (iv) We will use the two properties of f:

(III) $f(x) - x < 0$, for $x > \sqrt{2}$

(IV) $f(x) > \sqrt{2}$, for $x > \sqrt{2}$.

The proof is essentially the same as that of (I) and (II).

It suffices to prove that for every $n \geq 1$, the FINITE sequence $b_1, b_2, \ldots, b_n$ is decreasing and bounded below by $\sqrt{2}$.

Case $n=1$: $b_1 = a_2 = 1 + \frac{1}{1+\underbrace{a_1}_{=1}} = 1 + \frac{1}{2} > \sqrt{2}$.

Induction Step: Assume that the finite sequence $\{b_1, \ldots, b_n\}$ is decreasing and bounded below by $\sqrt{2}$. Then

$$b_{n+1} \underset{\substack{\uparrow \\ \text{Part (i)}}}{=} f(b_n) \underset{\substack{\uparrow \\ \boxed{\text{by (III) and the assumption that } \sqrt{2} < b_n}}}{<} b_n,$$

so $\{b_1, \ldots, b_n, b_{n+1}\}$ is decreasing. Furthermore

$$b_{n+1} = f(b_n) \underset{\substack{\uparrow \\ \boxed{\text{by (IV) and the assumption that } \sqrt{2} < b_n}}}{>} \sqrt{2},$$

so $\{b_1, \ldots, b_{n+1}\}$ is bounded below by $\sqrt{2}$.

Part ~~Prob~~ b(v): $\{b_n\}$ is monotonic decreasing and bounded below by (iv), so $\lim_{n\to\infty} b_n$ exist. Call it L. Then

$$L = \lim_{n\to\infty} b_n \underset{Q3(a)}{=} \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} f(b_n) \underset{\substack{f \text{ is continuous} \\ \text{on } (-3/2,\infty) \\ \text{and } L \geq \sqrt{2}}}{=} f\left(\lim_{n\to\infty} b_n\right) = f(L).$$

So $L = \sqrt{2}$, since $x = \sqrt{2}$ is the unique solution of $f(x) = x$ on $(0,\infty)$, by properties (I) and ($\boxed{\text{III}}$) of f.

The proof that $\lim_{n\to\infty} c_n = \sqrt{2}$ is completely analogous, using part (iii) instead of part (iv).

<u>CONCLUSION:</u>

We conclude that $\lim_{n\to\infty} a_n = \sqrt{2}$, by part (a) and part b(v).