Section 11.1 problem 76:

(a) Let $\lim_{n\to\infty} a_n = L$. Set $b_n = a_{n+1}$. In order to show that $\lim_{n\to\infty} b_n = L$,

We need to show: For every $\varepsilon > 0$ there exists N, such that $|b_n - L| \overset{*}{<} \varepsilon$ for all $n \geq N$. We remark that there exists $\tilde{N}$, such that $|a_n - L| < \varepsilon$, for all $n \geq \tilde{N}$, since $\lim_{n\to\infty} a_n = L$. Now $b_n = a_{n+1}$, and if $n \geq N$, then $n+1 \geq N$ and so (*) holds for the same N.

(b) Let $\lim_{n\to\infty} a_n = L$. We are given that the limit exists and $a_1 = 1$ and $a_{n+1} = \dfrac{1}{1+a_n}$. Hence $a_n \geq 0$ for all n.

so $L \geq 0$. Hence, $L = \lim_{n\to\infty} a_n \underset{\text{Part (a)}}{=} \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \dfrac{1}{1+a_n} =$

$$\frac{1}{1+\lim_{n\to\infty} a_n} = \frac{1}{1+L}.$$

$\boxed{\text{quotient rule and the fact that } L \neq -1, \text{ as } L \geq 0}$

Thus $L(1+L) = 1$, or equivalently

$$L^2 + L - 1 = 0$$

$$L \in \frac{-1 \pm \sqrt{1+4}}{2} = \left\{ \frac{-1+\sqrt{5}}{2}, \underbrace{\frac{-1-\sqrt{5}}{2}}_{<0} \right\} \text{ and } L \geq 0.$$

We conclude that $\boxed{L = \dfrac{-1+\sqrt{5}}{2}}$.