$$\sum_{n=0}^{\infty} c_n \left(x-\frac{\pi}{2}\right)^n$$

HW3 Q6: (a) Find the Taylor series of $\sin(x)$ centered at $a=\frac{\pi}{2}$ and its radius of convergence.

$$c_n = \frac{\sin^{(n)}\left(\frac{\pi}{2}\right)}{n!}$$

n	$\sin^{(n)}(x)$	$\sin^{(n)}\left(\frac{\pi}{2}\right)$	c_n	
0	$\sin(x)$	1	1	
1	$\cos(x)$	0	0	
2	$-\sin(x)$	-1	$-1/2!$	
3	$-\cos(x)$	0	0	
4	$\sin(x)$	1	$1/4!$	
⋮	periodic	⋮	⋮	
$n=2k$	$(-1)^k\sin(x)$	$(-1)^k$	$(-1)^k/(2k)!$	$=c_{2k}$
$n=2k+1$	$(-1)^k\cos(x)$	0	0	$=c_{2k+1}$

So, the Taylor series is $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$

Radius of convergence: (Using Quotient Rule)

Let $a_k = \dfrac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\left[\left(x-\frac{\pi}{2}\right)^{(2k+2)}/(2k+2)!\right]}{\left[\left(x-\frac{\pi}{2}\right)^{2k}/(2k)!\right]} =$$

$$= \lim_{k\to\infty}\frac{\left|x-\frac{\pi}{2}\right|^2}{(2k+1)(2k+2)} = 0 < 1, \text{ for all } x.$$

Hence, the series converges (absolutely), for all x and the radius of convergence is $+\infty$.

(b) Let $T_N(x) = \sum_{n=0}^{N} c_n \left(x-\frac{\pi}{2}\right)^n = \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(-1)^k}{(2k)!} \left(x-\frac{\pi}{2}\right)^{2k}$

where $\lfloor N/2 \rfloor$ is the integer part of $N/2$.

Let $E_N(x) := \sin(x) - T_N(x)$. We need to show that $\lim_{N\to\infty} |E_N(x)| = 0$.

Taylor's Error estimates states that

$$E_N(x) = \frac{\sin^{(N+1)}(c)}{(N+1)!} \left(x-\frac{\pi}{2}\right)^{N+1},$$ for some

point c between $\frac{\pi}{2}$ and x.

Now $|\sin^{(N+1)}(c)| \leq 1$, since $\sin^{(N+1)}(c)$ is $\pm\sin(c)$ or $\pm\cos(c)$. Hence

$$0 \overset{(+)}{\leq} |E_N(x)| \overset{(++)}{\leq} \frac{\left|x-\frac{\pi}{2}\right|^{N+1}}{(N+1)!}$$

Fix x.

There exists an integer N_0, (depending on x) such that $\frac{x-\frac{\pi}{2}}{N_0} < \frac{1}{2}$.

Then $\lim_{N\to\infty} \frac{\left|x-\frac{\pi}{2}\right|^{N+1}}{(N+1)!} = \lim_{N\to\infty} \frac{\left|x-\frac{\pi}{2}\right|^{N_0}}{N_0!} \cdot \underbrace{\frac{\left(x-\frac{\pi}{2}\right)}{(N_0+1)}}_{\wedge \, 1/2} \cdot \ldots \cdot \underbrace{\frac{\left(x-\frac{\pi}{2}\right)}{N+1}}_{\wedge \, 1/2} <$

$< \frac{\left|x-\frac{\pi}{2}\right|^{N_0}}{N_0!} \underbrace{\lim_{N\to\infty} \left(\frac{1}{2}\right)^{(N+1-N_0)}}_{0} = 0$. Hence, $\lim_{N\to\infty} |E_N(x)| = 0$,

by the squeeze Theorem (and (+) and (++)). $\square$