HW3 Q5;

(a) Show that $\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$ (*)

$$\frac{d}{dx}\tan^{-1}(x) = \underbrace{\frac{1}{1+x^2}}_{=\frac{1}{1-(-x^2)}} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

$\boxed{\text{for } |-x^2|<1 \iff |x|<1}$

Hence, over $(-1,1)$, integrating term by term, we get

$$\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

Now $\tan^{-1}(0) = 0 = C + 0$, so $C=0$.

Note that $\tan\left(\frac{\pi}{6}\right)$ belongs to $(-1,1)$, so

$$= \frac{\sin(\frac{\pi}{6})}{\cos(\frac{\pi}{6})} = \frac{(\frac{1}{2})}{(\frac{\sqrt{3}}{2})} = \frac{1}{\sqrt{3}}$$

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{(2n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n\sqrt{3}}$$

Multiplying both sides by $6 = 2(\sqrt{3})^2$ we get equation (*), as desired.

(b) Let $S_{11} = 2\sqrt{3}\sum_{n=0}^{10}\frac{(-1)^n}{(2n+1)3^n}$ and $R_{11} = 2\sqrt{3}\sum_{n=11}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$

We need to show that $|R_{11}| < 10^{-6}$.

The series (*) is an alternating series, $\lim_{n\to\infty}\frac{2\sqrt{3}}{(2n+1)3^n} = 0$, and the sequence $\left\{\frac{2\sqrt{3}}{(2n+1)3^n}\right\}_{n=0}^{\infty}$ is decreasing. The Remainder Estimate for alternating series states that

$|R_{11}| <$ Absolute value of first term in the remainder series $= \left|\frac{(-1)^{11}}{(23)3^{11}}\right| = \frac{1}{4{,}074{,}381} < 10^{-6}$ ∎