HW 3 Q3: $\sum_{n=1}^{\infty} c_n x^n$ has a radius of convergence 2

$\sum_{n=1}^{\infty} d_n x^n$ has a radius of convergence 3.

Then $\sum_{n=1}^{\infty} (c_n+d_n) x^n$ has a radius of convergence 2.

Proof: Let R be the radius of convergence of $\sum_{n=1}^{\infty} (c_n+d_n) x^n$. Then R is characterized by the property that the series converges for $|x|<R$ and diverges for $|x|>R$, by Theorem 4 in section 11.8. So we need to show that (a) $\sum_{n=1}^{\infty} (c_n+d_n) x^n$ converges, if $|x|<2$, and (b) $\sum_{n=1}^{\infty} (c_n+d_n) x^n$ diverges, if $|x|>2$.

(a) If $|x|<2$, then $|x|$ is smaller that the radii of $\sum_{n=1}^{\infty} c_n x^n$ and $\sum_{n=1}^{\infty} d_n x^n$ and so the latter two are convergent. Hence, so is $\sum_{n=1}^{\infty} \underbrace{c_n x^n + d_n x^n}_{\shortparallel}$.

(b) Assume that $|x|>2$.
We prove that $\sum_{n=1}^{\infty} (c_n+d_n) x^n$ diverges, by contradiction. Assume that it converges. Then $R \geq |x|$ and so $\sum_{n=1}^{\infty} (c_n+d_n) t^n$ converges for every t, such that $|t|<|x|$. Choose t, such that $2<|t|<\min\{3,|x|\}$. Then $\sum_{n=1}^{\infty} d_n t^n$ converges, since $|t|<3$, and $\sum_{n=1}^{\infty} (c_n+d_n) t^n$ converges, since $|t|<|x|$. Hence

$\sum_{n=1}^{\infty}[(c_n+d_n)-d_n]t^n$ converges. But the latter series is $\sum_{n=1}^{\infty} c_n t^n$ and it DIVERGES, since $|t| > 2$ = radius of convergence of $\sum_{n=1}^{\infty} c_n x^n$. A contradiction. Hence, $\sum_{n=1}^{\infty}(c_n+d_n)x^n$ diverges.

$\square$