HW3 problem 1:

(a) The series $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges for $p>1$ and diverges for $p \leq 1$.

Proof: The series converges if and only if the improper integral $I = \int_2^{\infty} \frac{1}{x(\ln(x))^p}\,dx$

converges, by the Integral test. Indeed, the function $f(x) = \frac{1}{x(\ln(x))^p}$ is positive

on $[2,\infty)$ and $f'(x) = (x^{-1}\ln(x)^{-p})' =$

$= -x^{-2}\ln(x)^{-p} - x^{-1}\left(-p\ln(x)^{-1-p}\cdot\frac{1}{x}\right) =$

$= \dfrac{-1+\frac{p}{\ln(x)}}{x^2\ln(x)^p} < 0$, for $x > e^p$,

and so $f(x)$ is decreasing on $[e^p, \infty)$.
Thus, the hypothese of the Integral Test are satisfied.

$I = \lim_{t\to\infty} \int_2^{t} \frac{1}{x(\ln(x))^p}\,dx \underset{\boxed{u=\ln(x)}}{=} \lim_{t\to\infty}\int_{\ln(2)}^{\ln(t)} \frac{1}{u^p}\,du = \lim_{u\to\infty}\int_{\ln(2)}^{u}\frac{1}{u^p}\,du$

$= \begin{cases} +\infty, & \text{if } p\leq 1 \\ \lim_{u\to\infty}\left[\frac{u^{1-p}}{1-p}\right]_{\ln(2)}^{u} = \frac{\ln(2)^{1-p}}{p-1}, & \text{if } p>1. \end{cases}$

Hence, the improper integral converges if and only if $p>1$. ☐

(b) $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^p}$ diverges for all p.

Proof: If $p \leq 0$, then $\lim_{n\to\infty} \frac{1}{(\ln(n))^p} =$

$$\lim_{n\to\infty} (\ln(n))^{|p|} = \left\{ \begin{array}{ll} 1, & \text{if } p=0 \\ +\infty, & \text{if } p<0 \end{array} \right\} \neq 0.$$

So, the series diverges for $p \leq 0$, by the n-th Term test.

Assume that $p > 0$. We will use the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n}$. The latter is the Harmonic series and it diverges. Hence to prove that $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^p}$ diverges, it suffices to prove that

$$\lim_{n\to\infty} \frac{\left[\frac{1}{(\ln(n))^p}\right]}{\left[\frac{1}{n}\right]} = +\infty.$$

Using the $\left[\frac{\infty}{\infty}\right]$ case of L'Hospital's Rule, we have

$$\lim_{n\to\infty} \frac{n}{(\ln(n))^p} = \lim_{x\to\infty} \frac{x}{(\ln(x))^p} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'Hop}}{=}}$$

$$= \lim_{x\to\infty} \frac{1}{p(\ln(x))^{p-1}\cdot\left(\frac{1}{x}\right)} = \left(\frac{1}{p}\right)\lim_{x\to\infty} \frac{x}{(\ln(x))^{p-1}} = \cdots$$

$\boxed{\text{Let } N \doteq \lceil p \rceil = \min\{z \text{ integer, such that } z \geq p\}. \text{ After } N \text{ uses of L'Hospital we get}}$

$$\cdots = \frac{1}{p(p-1)\cdots(p-N+1)} \lim_{x\to\infty} \frac{x}{(\ln(x))^{p-N}} = +\infty. \quad \square$$