Fall 2012 1321H Exam 2:

1) Partial Fraction. Skip.

2)(a) Use the comparison test to determine is the following is convergent or divergent:

$$\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx$$

Let us compare $\underbrace{\frac{\ln(x)}{x^2+1}}_{}$ $\overset{f(x)}{``}$ with $\frac{1}{x^{3/2}}$ $\overset{g(x)}{``}$.

We know that $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent, by the p-test, $p = 3/2 > 1$.

We need to show that

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ exists and is finite.

It would then follow that

$\int_1^\infty f(x)\,dx$ converges as well,

because both f, g are non-negative and $\int_1^\infty g(x)\,dx$ is

Let calculate

$$\lim_{x\to\infty} \frac{\ln(x)/x^2+1}{\left[1/x^{3/2}\right]} = \lim_{x\to\infty} \frac{\ln(x)\cdot x^{3/2}}{x^2+1} =$$

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}+x^{-3/2}} \underset{\left[\frac{\infty}{\infty}\right]}{\overset{\text{L'Hop}}{=}} \lim_{x\to\infty} \frac{(1/x)}{\frac{1}{2}x^{-1/2}+\left(-\frac{3}{2}\right)x^{-5/2}}$$

$$= \lim_{x\to\infty} \frac{1}{\underbrace{\frac{1}{2}x^{1/2}}_{\downarrow\ \infty} - \underbrace{\frac{3}{2}x^{-3/2}}_{\downarrow\ 0}} = \underset{\substack{\uparrow \\ \text{finite}}}{0} < \infty$$

Hence $\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx$ converges as well, by the comparison test.

(b) Evaluate the improper integral

$$\int_0^1 \frac{e^{\left(\frac{-1}{x}\right)}}{x^3}\,dx \overset{\text{def}}{=} \lim_{t\to 0^+} \int_t^1 \underbrace{\frac{e^{(-1/x)}}{x^3}}\,dx =$$

$u = \frac{1}{x}$ $u' = \frac{-1}{x^2}$

$$-u\,e^{-u}\,du = \left(-\frac{1}{x}\right)e^{-1/u}\left(-\frac{1}{x^2}\right)dx$$

$$= \lim_{t\to 0^+} -\int_{\underbrace{u(t)}_{1/t}}^{1} u e^{-u}\,du = \lim_{u\to\infty} -\int_{u}^{1} u e^{-u}\,du =$$

$$= \lim_{u\to\infty} \int_1^u u e^{-u}\,du \underset{\text{By parts}}{=} \lim_{u\to\infty}\left\{ \left[-u e^{-u}\right]_1^u + \int_1^u (\cancel{-}\,e^{-u})\,du \right\} =$$

By parts

$u = u \quad v' = e^{-u}$

$u' = 1 \quad v = -e^{-u}$

$$= \underbrace{-\underbrace{\left[\lim_{u\to\infty}\underbrace{\left(\frac{u}{e^u}\right)}_{\downarrow\, \left[\frac{\infty}{\infty}\right],\ \text{L'Hop}\ \to\ 0} - 1e^{-1}\right]}_{-e^{-1}}}_{e^{-1}} + \lim_{u\to\infty}\left[-e^{-u}\right]_1^u$$

$$\lim_{u\to\infty}\left[-e^{-u}\right]_1^u = \underbrace{-\left(\underbrace{\lim_{u\to\infty} e^{-u}}_{=0} - e^{-1}\right)}_{1/e}$$

$$= \boxed{\frac{2}{e}}.$$

3) a) Determine if the sequence

$\left\{ a_n = \boxed{\sqrt{n+2} - \sqrt{n}} \right\}_{n=1}^{\infty}$ converges or diverges.

Hint: $\boxed{(a-b)}(a+b) = a^2 - b^2$.

$$a_n = \sqrt{n+2} - \sqrt{n} = \frac{\left(\sqrt{n+2} - \sqrt{n}\right)\left(\sqrt{n+2} + \sqrt{n}\right)}{\sqrt{n+2} + \sqrt{n}} =$$

$$= \frac{(n+2) - n}{\sqrt{n+2} + \sqrt{n}} = \frac{2}{\sqrt{n+2} + \sqrt{n}} \xrightarrow{n \to \infty} 0.$$

(b) $\left\{ a_n = (n^2+3)^{1/n} \right\}_{n=1}^{\infty}$

$$\lim_{n \to \infty} (n^2+3)^{1/n} = \lim_{n \to \infty} e^{\ln\left((n^2+3)^{1/n}\right)} =$$

$$= \lim_{n \to \infty} e^{\left[\frac{\ln(n^2+3)}{n}\right]} = e^{\boxed{\lim_{n \to \infty} \frac{\ln(n^2+3)}{n}}} = L$$

Since e^x is a continuous function everywhere,

$$L = \lim_{n \to \infty} \frac{\ln(n^2+3)}{n} \overset{\text{L'Hop}}{\underset{\left[\frac{\infty}{\infty}\right]}{=}} \lim_{n \to \infty} \frac{[1/(n^2+3)] \cdot 2n}{1} =$$

Regard n now as a variable with real values

$$= \lim_{n \to \infty} \frac{2n}{n^2+3} = \lim_{n \to \infty} \frac{2}{\underset{\downarrow\, +\infty}{n} + \underset{\downarrow\, 0}{\left(\frac{3}{n}\right)}} = 0.$$

So $\lim_{n \to \infty} (n^2+3)^{1/n} = e^0 = \underline{1}$.

4) (a) Find the values of x, for which

$$\sum_{n=0}^{\infty} \underbrace{\frac{(2x-3)^n}{9^n}}_{a_n}$$ converges.

Note: $\frac{(2x-3)^n}{9^n} = \left(\frac{2}{9}\right)^n \left(x - 3/2\right)^n$

So the above is a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at $a = 3/2$ with $c_n = \left(\frac{2}{9}\right)^n$.

The series is a geometric series $\sum_{n=0}^{\infty} aR^n$ with radius $R = \frac{(2x-3)}{9}$.

We know that a geometric series converges, absolutely, to $\frac{a}{1-R}$, for $|R|<1$, and diverges, by the n-th term test, for $|R| \geq 1$.

So the series converges (absolutely)

for $\left|\frac{2x-3}{9}\right| < 1$ ← "OKAY answer".

$\iff$ $|2x-3| < 9$

$\left|x-\frac{3}{2}\right| < \frac{9}{2}$ ← better

$\underbrace{\frac{3}{2}-\frac{9}{2}}_{-3} < x < \underbrace{\frac{3}{2}+\frac{9}{2}}_{6}$ ← best.

$-3 < x < 6.$

(b) Find the sum.

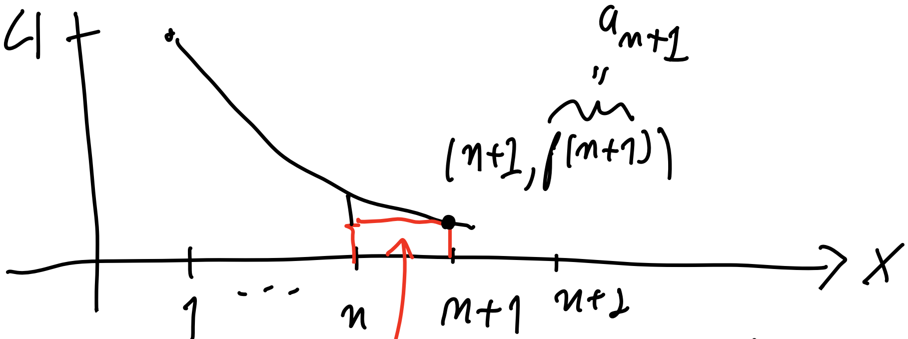
$$\sum_{n=0}^{\infty} \underbrace{\frac{(2x-3)^n}{9^n}}_{\left(\frac{2x-3}{9}\right)^n} \underset{\text{for } -3<x<6}{\overset{\uparrow}{=}} \frac{1}{1-\left(\frac{2x-3}{9}\right)} =$$

$$= \frac{9}{9-(2x-3)} = \frac{9}{12-2x}.$$

5) $s = \sum_{n=1}^{\infty} \frac{4}{n^5}$ and $s_n = \sum_{k=1}^{n} \frac{4}{k^5}$

Find the minimal number n, for which we know $|s - s_n| \le 10^{-8}$, by the error estimate of the Integral Test.

$y = \frac{4}{x^5} = f(x)$. $\quad a_n = \frac{4}{n^5} = f(n)$.



area $= a_{n+1} \overset{*}{<} \int_{n}^{n+1} f(x)\,dx$

$$\boxed{s - s_n = \sum_{k=n+1}^{\infty} a_k < \sum_{k=n+1}^{\infty} \int_{k-1}^{k} f(x)\,dx = \underbrace{\int_{n+1}^{\infty} f(x)\,dx}_{\text{error estimate}}}$$

So $0 < s - s_n < \int_{n+1}^{\infty} \frac{4}{x^5}\,dx = E_n$, by the Error Estimate of the Integral Test.

Note that $f(x) = \frac{4}{x^5}$ is positive and decreasing and $a_n = f(n)$, so the Integral Test applies.

$$E_n = \int_{n+1}^{\infty} \frac{4}{x^5}\,dx = \lim_{t\to\infty}\left[-x^{-4}\right]_{n+1}^{t} =$$

$$\overset{\text{def}}{=} \lim_{t\to\infty} \int_{n+1}^{t} \frac{4}{x^5}\,dx$$

$$= \lim_{t\to\infty} \underbrace{\frac{-1}{t^4}}_{\substack{\downarrow \\ 0}} - \left(\frac{-1}{(n+1)^4}\right) = \frac{1}{(n+1)^4}$$

$t \to \infty$

(+) WANT

$$10^{-8} = \frac{1}{(10^2)^4}$$

$n = 99$ is the smallest integer that satisfies the inequality.

$$(+) \Leftrightarrow (n+1)^4 \geq (100)^4 \Leftrightarrow n+1 \geq 100$$

$$\Leftrightarrow n \geq 99.$$

6) (a) Determine if abs conv. / Cond. conv. / divergent.

$$\sum_{n=1}^{\infty} \underbrace{\frac{\sqrt{n+10}}{n^2+3n+5}}_{a_n}$$

We will show that the series converges (and so absolutely conv, as it is a series of positive numbers) by the Limit Comparison Test with $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^{3/2}}}_{b_n}$. The series $\sum b_n$ converges, by the p-test $p = 3/2 > 1$. Hence, it suffices to show that

$L := \lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is finite,

in order to show that $\sum a_n$ converges as well.

$$L = \lim_{n\to\infty} \frac{\sqrt{n+10}}{n^2+3n+5} \cdot n^{3/2} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{(n+10)\, n^3}}{n^2+3n+5} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{10}{n}}}{1+\frac{3}{n}+\frac{5}{n^2}} \underset{\text{quotient rule}}{=}$$

$$= \frac{\lim_{n\to\infty} \sqrt{\quad = \quad}}{\lim_{n\to\infty} (\quad = \quad)} = \frac{\sqrt{1+0}}{1+0+0} = \frac{1}{1} < \infty.$$

Hence the series $\sum u_n$ converges (absolutely).

(b) $\displaystyle\sum_{n=1}^{\infty} \underbrace{(-1)^n e^{1/n}}_{a_n}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} e^{1/n} = e^{\lim_{n\to\infty} 1/n} = e^0 = 1 \neq 0$$

Hence, the series diverges, by the n-th Term Test.

(c) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}$

The series does NOT converge absolutely, in fact $\displaystyle\sum_{n=3}^{\infty} \underbrace{\frac{\ln(n)}{n}}_{|a_n|}$ diverges.

when $n \geq 3$ $\ln(n) > 1$, so

$$\underbrace{\frac{\ln(n)}{n}}_{|a_n|} > \underbrace{\frac{1}{n}}_{b_n}. \quad \text{Now } \sum b_n = \sum_{n=3}^{\infty} \frac{1}{n}$$

diverges by the p-test $p=1$ (Harmonic Series). So $\sum|a_n|$ diverges, by the Comparison Test.

We will use the Alternating Series Theorem (AST) to prove

$\boxed{\text{Conditional Convergence.}}$

$$\sum_{n\geq 3}^{\infty}(-1)^n \underbrace{\frac{\ln(n)}{n}}_{c_n}$$

We need to show:

(i) The sign is alternating $(c_n > 0)$

OKAY $\ln(n) > 0$, for $n \geq 3$

(ii) $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$

Indeed, $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} \underset{\left[\frac{\infty}{\infty}\right]}{\overset{\text{L'Hop}}{=}} \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0.$

(iii) The sequence $\{c_n\}$ is decreasing.

$c_n = \frac{\ln(n)}{n}$

It suffices to show that $f(x) = \frac{\ln(x)}{x}$ is decreasing on $[3, \infty)$. ($c_n = f(n)$.)

We will show that $f'(x) < 0$ on this interval.

$$f'(x) = \frac{1/x \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2} < 0$$

on this interval. ($\ln(x) > 1$ on $[3, \infty)$ because $\ln(3) > \ln(e) = 1$ and $\ln(x)$ is increasing, So for $x \geq 3$, $\ln(x) \geq \ln(3) > 1$

So $1 - \ln(x) < 0$ for $x \geq 3$.

Hence $\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}$ converges, by AST.

As it does NOT converges absolutely, we conclude

(d) that it is conditionally convergent.

$$\underbrace{\sum_{n=1}^{\infty} (-1)^n}_{a_n} \underbrace{\frac{n!\,2^n}{(2n)!}}_{b_n = |a_n|}$$

$$b_n = \frac{2^n}{(n+1)(n+2)\cdots(2n)} = \frac{1}{\left(\frac{n+1}{2}\right)\left(\frac{n+2}{2}\right)\cdots n}$$

Let us use the Ratio Test in order to prove absolute converges. We want to show that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

(the ineq needed to conclude abs convergence)

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{(n+1)}\, 2^{\cancel{n+1}}}{(2n+2)!} \cdot \frac{\cancel{(2n)!}^{\,1}}{\cancel{n!}_{\,1}\, \cancel{2^n}} =$$

$$(2n+2)! = \cancel{(2n)!}\,(2n+1)(2n+2)$$

$$= \lim_{n \to \infty} \frac{\cancel{(n+1)}^{1} \cdot 2}{(2n+1)\cancel{(2n+2)}_{2}} = \lim_{n \to \infty} \frac{1}{2n+2} =$$

$$= 0 < 1.$$

Hence, the series converges absolutely, by the Ratio Test.