Math 132H Hand written assignment 3 Fall 2023

- It is not sufficient to just write the answers. You must *explain* how you arrive at your answers showing all your work.

1. For what values of p do the following series converge? Prove your answer using the convergence/divergence tests we learned.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^p}$. Hint: For $p > 0$, use the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n}$

2. (Section 11.2 problem 93 page 750, modified, available on the webassign eBook) The Cantor set is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the previous step. The Cantor set consists of the numbers that remain in the interval $[0, 1]$ after all those intervals were removed.

   (a) Show that the total length of all the intervals that are removed is 1.

   (b) Let $\sum_{n=1}^{\infty} a_n \left(\frac{1}{3}\right)^n$ be a series, such that for every n the coefficient a_n is either 0 or 2. Show that the series converges and that its sum S belongs to the Cantor set. **Hint**: Let $S_N := \sum_{n=1}^{N} a_n \left(\frac{1}{3}\right)^n$ be the N-th partial sum and $R_N := \sum_{n=N+1}^{\infty} a_n \left(\frac{1}{3}\right)^n$ be the remainder. Set $S_0 = 0$. Show first that for every $N \geq 0$, S_N is a left end point of one of the intervals of length $(1/3)^N$ remaining after the N-th step. Then show that R_N either satisfies $0 \leq R_N \leq (1/3)^{N+1}$ or $2(1/3)^{N+1} \leq R_N \leq (1/3)^N$, so S does not belong to the interval $\left(S_N + \frac{1}{3^{N+1}}, S_N + \frac{2}{3^{N+1}}\right)$ that is removed in step $N + 1$.

   Note: In Math 300 you will learn that infinite sets come in different sizes using the notion of *cardinality*. Infinite sets of the same cardinality as the set of positive integers are said to be *countable*. These are the "smallest" infinite sets. The Cantor set is uncountably infinite, its cardinality is larger than that of the positive integers, while every infinite subset of the set of rational numbers is countable. In particular, the Cantor set contains irrational numbers (and not just end points of intervals that appear in the subdivision process, as those are all rational).

3. (Section 11.8 problem 45 page 786, available on the webassign eBook) Suppose that the series $\sum_{n=1}^{\infty} c_n x^n$ has a radius of convergence 2 and the series $\sum_{n=1}^{\infty} d_n x^n$ has a radius of convergence 3. What is the radius of convergence of $\sum_{n=1}^{\infty}(c_n + d_n)x^n$? **Hint**: Read carefully Theorem 4 in section 11.8. Note that if any two of the series $\sum_{n=1}^{\infty} c_n x^n$, $\sum_{n=1}^{\infty} d_n x^n$, and $\sum_{n=1}^{\infty}(c_n + d_n)x^n$ converge, so does the third.

4. (Section 11.8 problem 46 page 786, available on the webassign eBook). Suppose that the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^n$ is R. What is the radius of convergence of the power series $\sum_{n=1}^{\infty} c_n x^{2n}$?

5. (Section 11.9 problem 49 page 794, modified, available on the webassign eBook).

   (a) Use the power series of $\tan^{-1}(x)$ to prove the following expression for π as the sum of an infinite series:

   $$\pi = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

   (b) Show that the partial sum of the first 11 terms of the series in part 5a approximates π with an error less than 0.000001 (one over a million).

6. (Section 11.10 page 765 problems 27 and 32, modified).

   (a) Find the Taylor series of $\sin(x)$ centered at $a = \frac{\pi}{2}$ and find its radius of convergence.

   (b) Prove that the series in part 6a represents $\sin(x)$ for all x. Hint: Use Taylor's Inequality.