Math 132H Hand written assignment 2 Fall 2023

- It is not sufficient to just write the answers. You must *explain* how you arrive at your answers showing all your work.

1. (Section 7.4 problem 76 page 516, available on webassign eBook)
   (a) Use integration by parts to show that for every integer $n > 1$,
   $$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} \int \frac{dx}{(x^2+a^2)^{n-1}}$$
   Hint: Assume $n > 1$ and use integration by parts of $\int \frac{dx}{(x^2+a^2)^{n-1}}$ to prove the equality read from right to left. Note that $x^2 = (x^2+a^2) - a^2$.

   (b) Use part 1a to evaluate $\displaystyle\int \frac{dx}{(x^2+1)^2}$ and $\displaystyle\int \frac{dx}{(x^2+1)^3}$.

2. (Section 11.1 problem 53 page 736, available on webassign eBook) Determine if the sequence converges or diverges. If it converges, find the limit.
   $$a_n = \left(1 + \frac{2}{n}\right)^n$$
   Hint: Consider first the sequence $\ln(a_n)$. Note that $n\ln(x) = \frac{\ln(x)}{1/n}$.

3. (Section 11.1 problem 76 page 736, available on webassign eBook).
   (a) If a_n is convergent, show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.

   (b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$ for $n \geq 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

4. (Section 11.1 problem 85 page 736, available on webassign eBook). Let the sequence $\{a_n\}_{n=1}^{\infty}$ be defined by the rule $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. Show that the sequence is convergent, by carefully proving the hypothesis of the monotonic sequence theorem: (i) the sequence is bounded above by 2, (ii) the sequence is increasing. Finally compute the limit using the limit laws and theorems.

5. (Section 11.1 page 737 problem 98).
   (a) Use the precise definition of the limit of a sequence to prove:
   Theorem: If $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\lim_{n\to\infty} a_n = L$.

   (b) If $a_1 = 1$ and
   $$a_{n+1} = 1 + \frac{1}{1+a_n} \tag{1}$$
   find the first 8 terms of the sequence $\{a_n\}$. Then prove that $\lim_{n\to\infty} a_n = \sqrt{2}$. For this you will need to use part (a).

Observe that you provided in (b) the **continued fractions expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \frac{1}{2+\cdots}}$$

Hint for part (b): First prove that the sequence $\{a_n\}$ converges as follows. Let $b_n = a_{2n}$, $n = 1, 2, \ldots$ and $c_n = a_{2n+1}$, $n = 0, 1$, etc. It suffices to show that both sequences $\{b_n\}$ and $\{c_n\}$ converge to $\sqrt{2}$, by the above Theorem. Let $f(x) = (4+3x)/(3+2x)$.

(a) Show that $b_{n+1} = f(b_n)$ and $c_{n+1} = f(c_n)$. In other words, show that $f(a_{n+2}) = f(a_n)$. Simply use Equation (1) twice and take common denominators.

(b) Show that $f(x)$ is increasing over the positive x-axis and $f(\sqrt{2}) = \sqrt{2}$.

(c) Show that the sequence $\{c_n\}$ is increasing and bounded above by showing that for x in the interval $0 < x < \sqrt{2}$ the function f satisfies $f(x) - x > 0$ and $f(x) < \sqrt{2}$. Explain your work with complete sentences!

(d) Show that the sequence $\{b_n\}$ is decreasing and bounded below by showing that if x is larger than $\sqrt{2}$, the function f satisfies $f(x) - x < 0$ and $f(x) > \sqrt{2}$.

(e) Compute the limit of the sequences $\{b_n\}$ and $\{c_n\}$ using the limit laws and therems.