MATH 132 FALL 2009 FINAL EXAM

1. Evaluate the following integrals. Explicitly show any relevant algebraic manipulation.

   a)(7 points) $\int_0^2 (x^2+1)e^x dx$ $\Leftarrow$ integration by parts twice.

   b) (8 points) $\int \sqrt{1-4x^2}dx$ $\Leftarrow$ $2\sqrt{(\frac{1}{2})^2 - x^2}$ trig subs $x = \frac{1}{2}\sin\alpha$

2. (10 points) Find the volume of the infinite solid of revolution obtained by rotating the curve $y = \left(\frac{1}{x}\right)^{2/3}$ around the x-axis, over the interval $[1, \infty)$. Carefully justify your answer

3. (a) (5 points) Set-up a definite integral for the total length of the ellipse, given as the parametrized curve $x = 2\cos(\theta)$, $y = 3\sin(\theta)$, $0 \leq \theta \leq 2\pi$. Do **not** evaluate the integral.

   (b) (6 points) Sketch the region in the first quadrant that lies inside the polar curve $r = 2\sin(2\theta)$ and outside the polar curve $r = \sqrt{2}$. Provide polar coordinates for all points of intersection in the first quadrant.

   (c) (8 points) Determine the area of the region in part 3b.

4. a) (5 points) Find the derivative of the function $G(x) := \int_2^{1/x} \arctan(t)dt$

   b) (7 points) The velocity function, in meters per seconds, for a particle moving along a line is $v(t) = t^2 - 2t - 8$. Find the distance traveled by the particle (**not** the displacement) during the time interval $2 \leq t \leq 6$.

5. Determine whether the following series converge absolutely, converge conditionally, or diverge. Name each test you use and indicate why all the conditions needed for it to apply actually hold.

   (a) (7 points) $\sum_{n=1}^{\infty}(-1)^n \frac{2n^2}{7n^4 - n^3 + 1}$

   (b) (7 points) $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n(\ln(n))^2}$

6. a) (7 points) Find the Maclaurin series for $f(x) = \ln(1+3x)$.

   b) (8 points) Determine the **interval** of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n}$. Justify your answer with calculations. Do not forget to check for convergence at the end points.

7. a) (7 points) Let $f(x) = e^x + e^{-x}$ and denote by $T_n(x)$ its Taylor polynomial, centered at 0, involving powers of x of degree $\leq n$. Find the Taylor polynomial $T_7(x)$.

   b) (8 points) Use Taylor's Inequality to show that the error $|f(x) - T_7(x)|$, of approximating $f(x)$ by $T_7(x)$, is bounded by 0.0001 over the interval
   $-1 \leq x \leq 1$. Carefully justify your answer!

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b) (8 points) Use Taylor's Inequality to show that the error $|f(x) - T_7(x)|$, of approximating $f(x)$ by $T_7(x)$, is bounded by 0.0001 over the interval $-1 \le x \le 1$. Carefully justify your answer! $\simeq 10^{-4}$

a) Recall that the Taylor series of f centered at 0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$ and

$$T_{N}(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \qquad N = 7$$

n	0	1	2	3	4	...	$2k$	$2k+1$
$f^{(n)}(x)$	e^x+e^{-x}	e^x-e^{-x}	e^x+e^{-x}	e^x-e^{-x}	e^x+e^{-x}		e^x+e^{-x}	e^x-e^{-x}
$f^{(n)}(0)$	2	0	2	0	2		2	0
$\frac{f^{(n)}(0)}{n!}$	2	0	$2/2!$	0	$2/4!$		$\frac{2}{(2k)!}$	0

$$T(x) = 2 + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{2}{4!}x^4 + 0x^5 + \frac{2}{6!}x^6 + 0x^7 + \dots$$

$$\boxed{T_7 = 2 + x^2 + \frac{1}{2}x^4 + \frac{2}{6!}x^6}$$

$$T(x) = \sum_{k=0}^{\infty} \frac{2}{(2k)!} x^{2k}$$

b) Show that $\left| f(x) - T_7(x) \right| \leq 10^{-4}$, for $-1 \leq x \leq 1$.

Taylor's Error Estimate: There exists c between 0 and x,

$$f(x) - T_7(x) \overset{(*)}{=} \frac{f^{(8)}(c)}{8!} x^8$$

$$\boxed{f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}} \qquad \boxed{n=7}$$

If $\left| f^{(8)}(x) \right| \leq M$, for all x in $[-1,1]$,

then $\left| f(x) - T_7(x) \right| \leq \frac{M}{8!} |x|^8$, by $(*)$

Taylor's Inequality.

$$f^{(8)}(x) = f(x) = e^x + e^{-x} = \underset{\underset{e^1}{\wedge|}}{e^{|x|}} + \underset{\underset{e^0}{\wedge|}}{e^{-|x|}} \leq \overset{M}{\overset{\shortparallel}{e+1}} \leftarrow \text{for } -1 \leq x \leq 1$$

$$|f(x)-T_7(x)| \leq \frac{e+1}{8!}|x|^8 \underset{\substack{\uparrow \\ \text{for} \\ -1\leq x\leq 1}}{\leq} \frac{e+1}{8!} = 9.22\ldots \cdot 10^{-5} < \frac{1}{10^4}$$

as desired.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad 59$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$e^x + e^{-x} = \sum_{k=0}^{\infty} 2 \cdot \frac{x^{2k}}{(2k)!}$$

6. a) (7 points) Find the Maclaurin series for $f(x) = \ln(1+3x)$.

b) (8 points) Determine the **interval** of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n}$. Justify your answer with calculations. Do not forget to check for convergence at the end points.

a) $f(x) = \ln(1+3x)$

MacLaurin Series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

(Taylor series centered at 0).

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\ln(1+3x)$	$\ln(1)=0$	0
1	$\frac{3}{1+3x}$	3	3
2	$\frac{-3^2}{(1+3x)^2}$	-3^2	$-3^2/2!$
3	$\frac{2\cdot 3^3}{(1+3x)^3}$	$2\cdot 3^3$	$2\cdot 3^3 / 3! = 3^2$
4	$\frac{-2\cdot 3\cdot 3^4}{(1+3x)^4}$	$\frac{(-1)^{4+1}\cdot 3!\cdot 3^4}{(1+3x)^4}$	

power

chain Rule

$$\frac{d}{dx}\left(2\cdot 3^3(1+3x)^{-3}\right) = 2\cdot 3^3\cdot(-3)\,(1+3x)^{-4}\cdot 3$$

$$= -2\cdot 3\cdot 3^4\,(1+3x)^{-4}$$

power

$\vdots$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
n	$\frac{(-1)^{n+1}(n-1)!\,3^n}{(1+3x)^n}$	$(-1)^{n+1}(n-1)!\,3^n$	$\frac{(-1)^{n+1}\cdot 3^n}{n}$

for $n \geq 1$

$\frac{(-1)^{n+1}\cdot 3^n}{n} = f^{(n)}(0)/n!$

$$\underbrace{T(x)}_{\text{Maclaurin Series}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 3^n \, x^n$$

Method II: $f(x) = \ln(1+3x)$.

Find the Maclaurin Series of $\ln(1+u)$ in powers of u and plug in $u = 3x$.

for $|u|<1$

$$\frac{\partial}{\partial u} \ln(1+u) = \frac{1}{1+u} = \frac{1}{1-(-u)} = \sum_{n=0}^{\infty} (-u)^n =$$

$$= \sum_{n=0}^{\infty} (-1)^n u^n$$

Integrating term by term

$$\ln(1+u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{n+1}}{n+1} + \underbrace{C}_{=0}$$

$$\boxed{k=n+1} \quad = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{u^k}{k}$$

Plug in $u = 3x$

$$\ln(1+3x) \underset{|3x|<1}{=} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(3x)^k}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k x^k}{k}$$

$n=2$

6. a) (7 points) Find the Maclaurin series for $f(x) = \ln(1+3x)$.

   b) (8 points) Determine the **interval** of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n}$. Justify your answer with calculations. Do not forget to check for convergence at the end points.

b) Ratio Test:

$$\underset{\text{need}}{1 >} \lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{n+1} \right| \Big/ \left| \frac{(-2)^n x^n}{n} \right| =$$

$$\lim_{n\to\infty} \frac{2^{\cancel{n}+1} |x|^{(\cancel{n}+1)} \cdot n}{\underset{1}{\cancel{2^n}}\ \underset{1}{\cancel{|x|^n}}\ (n+1)} = \lim_{n\to\infty} |2x| \underbrace{\left(\frac{n}{n+1}\right)}_{\xrightarrow{n\to\infty} 1} =$$

$$= |2x|.$$

The series converges absolutely for $|2x| < 1$ ($|x| < \frac{1}{2}$) and diverges for $|x| > \frac{1}{2}$, so the radius of convergence is $R = \frac{1}{2}$.

At $x = \frac{1}{2}$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} (-1)^n / n$$

converges by the AST

$1/n$ is decreasing, the sign is alternating, and $\lim_{n \to \infty} \frac{1}{n} = 0$.

At $x = -\frac{1}{2}$, the series is

$$\sum_{n=1}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

by the p-test, $p = 1$ not larger than 1.

Interval of convergence is

$\left(-\frac{1}{2}, \frac{1}{2}\right]$, $\quad -\frac{1}{2} < x \leq \frac{1}{2}$.

5. Determine whether the following series converge absolutely, converge conditionally, or diverge. Name each test you use and indicate why all the conditions needed for it to apply actually hold.

(a) (7 points) $\sum_{n=1}^{\infty} (-1)^n \frac{2n^2}{7n^4 - n^3 + 1}$ a_n

(b) (7 points) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2}$

(a) Let $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (absolutely)

b_n

Use the Limit comparison Test

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{2n^2}{7n^4 - n^3 + 1} \cdot n^2 =$$

$$= \lim_{n\to\infty} \frac{2}{7 - \underbrace{\frac{1}{n}}_{\downarrow n\to\infty \; 0} + \underbrace{\frac{1}{n^4}}_{\downarrow \; 0}} = \frac{2}{7} \text{ finite } < \infty$$

So $\sum a_n$ converges absolutely.

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{n(\ln(n))^2}$

Let us test for absolute convergence

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}.$$ We will use the integral test. The series converges (absolutely) if and only if

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx$$ converges

because $f(x) = \dfrac{1}{x(\ln(x))^2}$ is positive and decreasing on $[2,\infty)$.

$$\lim_{t\to\infty} \int_2^{t} \underbrace{\frac{1}{x(\ln(x))^2}\,dx}_{\frac{1}{u^2}\,du} =$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{u=\ln(t)} \frac{1}{u^2}\,du = \lim_{u\to\infty} \Big[-u^{-1}\Big]_{\ln(2)}^{u} =$$

$$\lim_{u\to\infty}$$

$$= \lim_{u\to\infty} \left(-\frac{1}{u} - \left(-\frac{1}{\ln(2)}\right)\right) = \frac{1}{\ln(2)} < \infty$$

So the improper integral converges, hence so does the series.

So the original series $\boxed{\text{Converges absolutely}}$.

4. a) (5 points) Find the derivative of the function $G(x) := \int_2^{1/x} \arctan(t)dt$

b) (7 points) The velocity function, in meters per seconds, for a particle moving along a line is $v(t) = t^2 - 2t - 8$. Find the distance traveled by the particle (**not** the displacement) during the time interval $2 \leq t \leq 6$.

a) $G(x) = \int_2^{1/x} \arctan(t)\,dt$

$= F(1/x)$, where $F(u) = \int_2^{u} \arctan(t)dt$

$\frac{\partial}{\partial x} F(1/x) = \frac{\partial}{\partial u} F(u) \cdot \underbrace{\frac{\partial u}{\partial x}}_{-\frac{1}{x^2}}$

Chain Rule

$u = 1/x$

$\frac{\partial}{\partial u} F(u) = \arctan(u)$

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$\left(\arctan\left(\frac{1}{x}\right)\right) \cdot \left(-\frac{1}{x^2}\right).$

b) $v(t) = t^2 - 2t - 8, \quad 2 \le t \le 6$

distance traveled = integral of the speed $|v(t)|$.

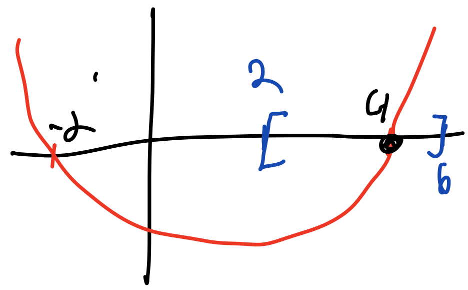
$$\int_2^6 |v(t)|\,dt$$

$$0 = v(t) = t^2 - 2t - 8$$

$$\frac{-(-2) \pm \sqrt{(-2)^2 - 4(1\cdot(-8))}}{2} =$$

$$= 1 \pm \sqrt{1+8} =$$

$$= 1 \pm 3 = 4, -2.$$



$v(4) = 0, \quad v(2) = -10$

$v(6) =$

$$|v(t)| = \begin{cases} -v(t), & 2 \le t \le 4 \\ v(t), & 4 \le t \le 6 \end{cases}$$

$$\text{distance} = \int_2^4 -(t^2 - 2t - 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt =$$

$= \ldots$

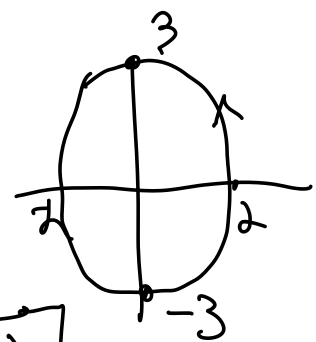
answer

3. (a) (5 points) Set-up a definite integral for the total length of the ellipse, given as the parametrized curve $x = 2\cos(\theta)$, $y = 3\sin(\theta)$, $0 \le \theta \le 2\pi$. Do **not** evaluate the integral.

(b) (6 points) Sketch the region in the first quadrant that lies inside the polar curve $r = 2\sin(2\theta)$ and outside the polar curve $r = \sqrt{2}$. Provide polar coordinates for all points of intersection in the first quadrant.

(c) (8 points) Determine the area of the region in part 3b.

(a) $x = 2\cos(\theta)$, $y = 3\sin(\theta)$, $0 \le \theta \le 2\pi$,



$$L = \text{Length} = \int_0^{2\pi} \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2}\, d\theta =$$

arclength for a parametrized curve

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta} 2\cos(\theta) = -2\sin(\theta)$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta} 3\sin(\theta) = 3\cos(\theta).$$

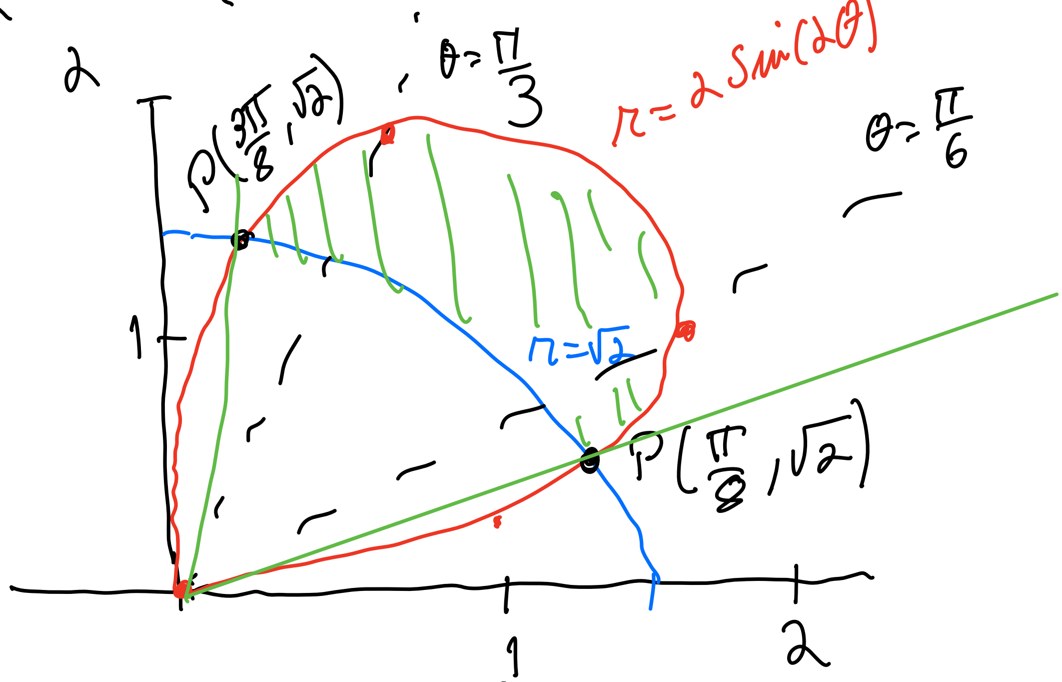
$$L = \int_0^{2\pi} \sqrt{4\cos^2(\theta) + 9\sin^2(\theta)}\, d\theta$$

b) Region $r = 2\sin(2\theta)$ ← inside first quadrant

$r = \sqrt{2}$ ← outside. ← $\boxed{\text{circle of radius } \sqrt{2} \text{ centered at } (0,0).}$

and in the first quadrant,

θ	$2\sin(2\theta)$
0	0
$\pi/12$	$2\sin\left(\frac{\pi}{6}\right) = 1$
$\pi/6$	$2\sin(\pi/3) = 2\,\sqrt{3}/2 = \sqrt{3}$
$\pi/3$	$2\sin(2\pi/3) = 2\sin\left(\pi - \frac{\pi}{3}\right) = 2\sin\left(\pi/3\right) = \sqrt{3}$
$\frac{\pi}{2}$	$2\sin(\pi) = 0$



Points of intersection:

$$\sqrt{2} = r = 2\sin(2\theta)$$

$$\sin(2\theta) = \frac{\sqrt{2}}{2} = \sin\left(\frac{\pi}{4}\right) = \sin\left(\underbrace{\pi - \frac{\pi}{4}}_{\frac{3\pi}{4}}\right)$$

$$\boxed{\theta = \frac{\pi}{8}, \frac{3\pi}{8}}$$

$$r = \sqrt{2}$$

Area of the region:

$$\int_{\pi/8}^{3\pi/8} \frac{1}{2}\left[R^2(\theta) - r^2(\theta)\right] d\theta =$$

R = large radius $= 2\sin(2\theta)$

r = small radius $= \sqrt{2}$

$$= \frac{1}{2}\int_{\pi/8}^{3\pi/8} \left[4\underbrace{\sin^2(2\theta)}_{\frac{1-\cos(4\theta)}{2}} - 2\right] d\theta =$$

$$= \frac{1}{2}\int_{\pi/8}^{3\pi/8} \cancel{2} - 2\cos(4\theta) \cancel{-2}\, d\theta$$

$$= -\int_{\pi/8}^{3\pi/8} \cos(4\theta)\, d\theta = -\left[\frac{1}{4} \sin(4\theta) \right]_{\pi/8}^{3\pi/8}$$

$$= -\frac{1}{4}\left(\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{-1} - \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} \right) = \boxed{\frac{1}{2}}$$