1. (8) Evaluate $\int 5e^{2x} - \frac{1+\sqrt{x}}{3x} + \frac{3}{1+x^2} dx$

2. (8) Evaluate $\int \sin^5(x)\cos^4(x) dx$

3. (8) Evaluate $\int x^3 \sin(x^2) dx$

4. (8) Evaluate $\int \sqrt{9-x^2} dx$

1) $\int 5e^{2x} - \frac{1+\sqrt{x}}{3x} + \frac{3}{1+x2} dx =$

$$= 5\underbrace{\int e^{2x} dx}_{\overset{\shortparallel}{\frac{1}{2}e^{2x}}} - \frac{1}{3}\underbrace{\int \frac{1}{x} dx}_{\ln|x|} - \frac{1}{3}\underbrace{\int \frac{1}{\sqrt{x}} dx}_{2\cdot x^{\frac{1}{2}}} + 3\underbrace{\int \frac{1}{1+x^2} dx}_{\tan^{-1}(x)}$$

$$= \frac{5}{2} e^{2x} - \frac{1}{3}\ln|x| - \frac{2}{3} x^{1/2} + 3\tan^{-1}(x) + C$$

2. (8) Evaluate $\underbrace{\int \sin^5(x)\cos^4(x)dx}_{\left[\underbrace{\sin^4(x)}_{\substack{(\sin^2(x))^2 \\ (1-\cos^2(x))^2}} \cos^4(x)\right]\sin(x)} =$

$$= \int (1-\cos^2(x))^2 \cos^4(x) \cdot \underbrace{\sin(x)dx}_{-du} =$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$$= -\int \underbrace{(1-u^2)^2 u^4}_{(1-2u^2+u^4)u^4} \quad du$$

$$= -\int u^8 - 2u^6 + u^4\,du =$$

$$= -\left[\frac{u^9}{9} - \left(\frac{2}{7}\right)u^7 + \frac{u^5}{5}\right. + C =$$

$$= -\left[\frac{(\cos(u))^9}{9} - \left(\frac{2}{7}\right)(\cos(u))^7 + \frac{(\cos(u))^5}{5}\right] + C_1 .$$

3. (8) Evaluate $\int x^3 \sin(x^2)dx =$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}}$$

$$\int \underbrace{x^2}_{u} \sin(\overbrace{x^2}^{u}) \underbrace{x\,dx}_{\frac{1}{2}du} = \frac{1}{2}\int u \sin(u)\,du =$$

$u = u \quad v' = \sin(u)$

$u' = 1 \quad v = -\cos(u)$

$$= \frac{1}{2}\left[-u\cos(u) + \underbrace{\int (+\cos(u))\,du}_{\sin(u)}\right] + C =$$

$$= \frac{1}{2}\left[-x^2\cos(x^2) + \sin(x^2)\right] + C_1$$

$\underline{\text{Check:}}$ $\frac{d}{dx}(\;''\;) = \frac{1}{2}\left[-\left(\cancel{2x\cos(x^2)} + x^2(-\sin(x^2)2x)\right) + \cancel{\cos(x^2)2x}\right]$

$$= \frac{1}{2} x^3 \sin(x^2).$$

4. (8) Evaluate $\int \sqrt{9 - x^2}\,dx$

$\sqrt{\underbrace{a^2}_{9} - x^2}$ use inverse trig subst.

$\boxed{x = \underbrace{a}_{3} \sin(\theta)}$; $\frac{dx}{d\theta} = 3\cos(\theta)$

$$= \int \sqrt{\underbrace{9 - (3\sin(\theta))^2}_{9[\underbrace{1 - \sin^2(\theta)}_{\cos^2(\theta)}]}}\; 3\cos(\theta)\,d\theta =$$

$\cos(\theta) > 0$

$$= \int \underbrace{3\cos(\theta) \cdot 3\cos(\theta)}_{9 \underbrace{\cos^2(\theta)}_{\frac{1 + \cos(2\theta)}{2}}}\, d\theta =$$

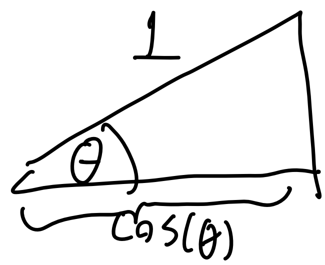
$$= 9\int \frac{1+\cos(2\theta)}{2}\,d\theta =$$

$$= 9\left[\frac{\theta}{2} + \frac{1}{2^2}\overbrace{\sin(2\theta)}^{2\overbrace{\sin(\theta)}^{= x/3}\cos(\theta)}\right] + C =$$

$$= 9\left[\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{4}\cdot 2\left(\frac{x}{3}\right)\underbrace{\cos(\theta)}_{= \sqrt{1-\left(\frac{x}{3}\right)^2}}\right] + C$$

$x = 3\sin(\theta)$

$\sin^{-1}\left(\frac{x}{3}\right) = \theta$



$$\frac{x}{3} = \sin(\theta)$$

$$= \sqrt{1-\left(\frac{x}{3}\right)^2}$$

$$= 9\left[\sin^{-1}\left(\frac{x}{3}\right) + \frac{1}{2}\left(\frac{x}{3}\right)\sqrt{1-\left(\frac{x}{3}\right)^2}\right] + C$$

4 (15) Determine the following derivatives. Briefly justify each answer.

a) $\frac{\partial}{\partial x}\int_1^{x^2} \sin(t+t^2)dt$.

b) $\frac{\partial}{\partial x}\int_x^{10} \ln(1+t^2)dt$

c) $\frac{\partial}{\partial x}\int_0^{\pi} \sin(x)dx$

$\frac{\partial}{\partial x}(\text{constant}) = 0.$

a) $\frac{\partial}{\partial x}\int_1^{x^2} \sin(t+t^2)\,dt \underset{\text{chain Rule}}{=}$

$x^2 = u(x)$

$$\underbrace{\left(\frac{\partial}{\partial u}\int_1^{u} \sin(t+t^2)\,dt\right)}_{\text{// F.T.C}} \cdot \underbrace{\left(\frac{\partial u}{\partial x}\right)}_{2x} =$$

$$\sin(\underbrace{u}_{x^2} + \underbrace{u^2}_{(x^2)^2})$$

$$= \sin(x^2 + x^4)\cdot 2x.$$

b) $\frac{\partial}{\partial x}\int_x^{10} \ln(1+t^2)\,dt \overset{def}{=} \frac{\partial}{\partial x}\left(-\int_{10}^{x} \ln(1+t^2)\,dt\right)$

$$\underset{\text{F.T.C}}{=} -\ln(1+x^2).$$

5 (15) Integrate $\displaystyle\int \frac{\overbrace{x^4+x^3+x-2}^{N(x)}}{\underbrace{x^4+x^2}_{D(x)}}\,dx$

Write $N(x) = Q(x)D(x) + R(x)$, where $R(x)$, $Q(x)$ are poly and $\deg R < \deg D$

$$(x^4+x^3+x-2) = \boxed{1}(x^4+x^2) + \boxed{}$$

$$\begin{array}{r} 1 \\ (x^4+x^2)\overline{)\,x^4+x^3+x-2} \end{array}$$

$$\boxed{(x^4+x^3+x-2)-(x^4+x^2) = x^3-x^2+x-2}$$

$$\frac{x^4+x^3+x-2}{x^4+x^2} = 1 + \frac{x^3-x^2+x-2}{\underbrace{x^4+x^2}_{x^2(x^2+1)}}$$

Partial Fractions:

$$\frac{x^3-x^2+x-2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{c_1 x + c_0}{x^2+1}$$

So

$$\int 1 + \frac{x^3-x^2+x-2}{x^4+x^2}\, dx =$$

$$\int 1 + \frac{A}{x} + \frac{B}{x^2} + \frac{c_1 x + c_0}{x^2+2}\, dx =$$

$$= x + A\ln|x| + \frac{(-B)}{x} + \underbrace{\int \frac{\overset{=\left(\frac{c_1}{2}\right)du}{c_1 x}}{x^2+1}\, dx}_{\substack{u=(x^2+1) \\ du = 2x\,dx}} + \underbrace{\int \frac{c_0}{x^2+1}\, dx}_{c_0 \tan^{-1}(x)}$$

$$\frac{c_1}{2}\int \frac{du}{u}$$

$$\frac{c_1}{2}\ln(x^2+1)$$

Common denominators:

$$\frac{x^3 - x^2 + x - 2}{x^2(x^2+1)} = \frac{Ax(x^2+1) + B(x^2+1) + (C_1x+C_0)x^2}{x^2(x^2+1)}$$

$$x^3 - x^2 + x - 2 = Ax(x^2+1) + B(x^2+1) + (C_1x+C_0)x^2$$

Finding the constants A, B, C_1, C_0.

when $x = 0$

$$\boxed{-2 = B}$$

when $x = 1$

$$-1 = 2A + \overset{-4}{\overbrace{2\underbrace{B}_{-2}}} + C_1 + C_0 \quad (*)$$

when $x = -1$

$$-5 = -2A + \underbrace{2B}_{=-4} - C_1 + C_0 \quad (**)$$

Adding the last two

$$-6 = -8 + 2C_0 \qquad \boxed{C_0 = 1}$$

Taking the difference of $(*)$, $(**)$

$4 = 4A + 2C_1$

No need to finish, ...

$\boxed{A = 1}$ $\boxed{C_1 = 0}$ $\boxed{B = -2}$ $\boxed{C_1 = 1}$

6 (15) a) Approximate the integral $\int_0^1 e^{(x^2)}dx$ by a Riemann sum that uses 5 equal-length sub-intervals and **left**-hand endpoints as sample points. (Show the individual terms of the Riemann sum before you calculate the value of the sum).

b) Interpret the limit $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1-\left(\frac{i}{n}\right)^2} \cdot \left(\frac{1}{n}\right)$ as the area of a region. Justify this interpretation! Graph this region. Use your interpretation to evaluate the limit.

a)

$x_0 = 0$, $0.2 = x_1$, $0.4 = x_2$, $0.6 = x_3$, $0.8 = x_4$, $1 = x_5$

$\Delta x = \frac{1}{5}$

Riemann Sum:

$$\underbrace{e^{(0^2)}}_{e^{(x_0^2)}} \cdot \underbrace{\frac{1}{5}}_{\Delta x} + \underbrace{e^{(0.2^2)}}_{e^{(x_1^2)}} \cdot \underbrace{\frac{1}{5}}_{\Delta x} + e^{(0.4^2)} \cdot \frac{1}{5} + e^{(0.6)^2} \cdot \frac{1}{4} + e^{(0.8)^2} \cdot \frac{1}{5}$$

--- use calculator.

b) $$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1-\Big(\underbrace{\tfrac{i}{n}}_{x_i^*}\Big)^2}\;\underbrace{\left(\frac{1}{n}\right)}_{\Delta x} = \int_0^1 \sqrt{1-x^2}\,dx$$

$[a, b]$, $a = 0$, $b = 1$

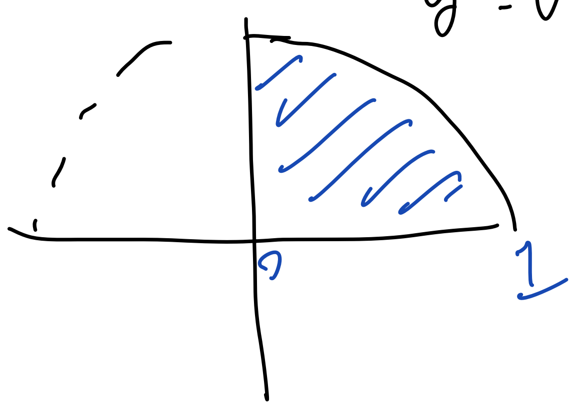
$\Delta x = ?$

$x_i^* = ? \quad = \frac{i}{n}$

$$\sum_{i=1}^{n} \underbrace{f(x_i^*)}_{\sqrt{1-(x_i^*)^2}} \Delta x$$

$$\underbrace{f(x)}_{=\sqrt{1-x^2}} = ?$$

$y = \sqrt{1-x^2}$

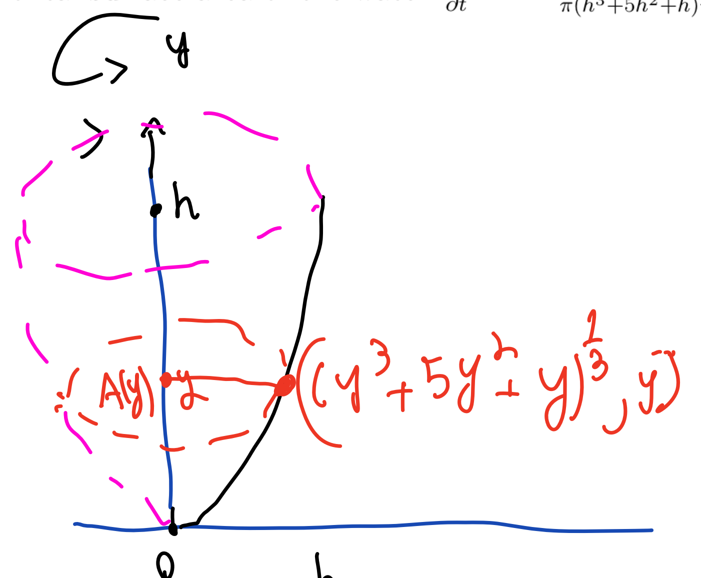


$$\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

7 (15) The graph of the curve $x = (y^3 + 5y^2 + y)^{\frac{1}{3}}$, $y > 0$, is revolved about the y-axis to form the outer surface of a water container. Water is being poured in at a constant rate of 10 centimeters cubed per second. Assume the x and y units are in centimeters.

a) Use the disk (washer) method to set-up an integral for the volume $V(h)$ of the water in the container, when the water level reaches the horizontal line $y = h_{\text{cm}}$ (for some constant height h). Do **NOT** evaluate the integral.

b) Use the fact that $\frac{\partial V(h(t))}{\partial t} = 10_{\text{cm}^3/\text{sec}}$, the Fundamental Theorem of Calculus, and the Chain Rule, to show that the rate of change $\frac{\partial h}{\partial t}$, of the height $h(t)$ of the water in the container with respect to time, is equal to 10 divided by the horizontal-surface-area of the water $\frac{\partial h}{\partial t} = \frac{10}{\pi(h^3+5h^2+h)^{2/3}}$ cm/sec.



$$\text{a)} \quad V(h) = \int_0^h \underbrace{A(y)}_{= \pi(\underbrace{R(y)}_{=})^2} \, dy$$

$\leftarrow$ the x-coordinate of the pt on the curve with y-coor $= y$

$$R(y) = (y^3 + 5y^2 + y)^{1/3}$$

$$V(h) = \pi \int_0^h \left((y^3 + 5y^2 + y)^{1/3} \right)^2 dy$$

b) Relate the rate of change of the volume $\frac{\partial}{\partial t} V(h(t))$

to $\frac{\partial}{\partial t} h(t)$.

Express $\frac{\partial h}{\partial t}$ in terms of $\frac{\partial}{\partial t} V(h(t))$ and ("surface area")

10

|| at the given time t_0

$$\frac{\partial}{\partial t} V(h(t)) = \frac{\partial}{\partial t} \pi \int_0^{h(t)} (y^3 + 5y^2 + y)^{2/3} dy =$$

$$\underset{\text{F.T.C}}{=} \pi \left(h(t)^3 + 5h(t)^2 + h(t) \right)^{2/3} \underset{\text{Chain Rule}}{\cdot} \frac{\partial h}{\partial t}$$

$$\frac{\partial h}{\partial t} = \frac{\boxed{\frac{\partial}{\partial t} V(h(t))} = 10}{\boxed{\pi \left(h(t)^3 + 5h(t)^2 + h(t)\right)^{2/3}}}$$

$= A(h(t))$

Area of the surface of the water at height $h(t)$,

8 (16) A ball is thrown upward from a tower window, 200 feet above the ground, with initial velocity $v_0 = 48$ feet per second. Its acceleration, t seconds afterwards, is $v'(t) = -32$ ft/sec^2.

a) Find the velocity $v(t)$ of the ball t seconds after it is thrown, but before it hits the ground.

b) Find the height $h(t)$ of the ball above the ground t seconds after it is thrown.

c) Find the total distance traveled by the ball during the time interval $0 \le t \le 2$ seconds.

$v_0 = 48$ ft/sec

$v'(t) = -32$ ft/sec^2

a) $v(t) = \int -32\, dt + C_1 = -32t + C_1$

$v(0) = 48 = -32 \cdot 0 + C_1$. So $C = 48$

$v(t) = -32t + 48.$

b) $\left(h(t)-\underbrace{h(0)}_{200}\right) \underset{FTC}{=} \int_0^t v(t)\,dt = \int_0^t -32t+48\,dt =$

$h'(t)=v(t)$

$$= \left[-16t^2+48t\right]_0^t =$$

$$= -16t^2+48t$$

$$\boxed{h(t) = -16t^2+48t+200}$$

$$h(2) = -16\cdot 4+48\cdot 2+200 > 0$$

Total distance =

$$\int_0^2 \underbrace{|v(t)|}_{\text{speed}}\,dt = \int_0^{3/2} \overbrace{v(t)}^{48-32t}\,dt + \int_{3/2}^{2} \overbrace{-v(t)}^{32t-48}\,dt$$

$$\boxed{\begin{array}{l} v(t) = -32t+48, = 0 \\ v(t)=0 \text{ when } t = \frac{48}{32} = \frac{3}{2} = 1\frac{1}{2} \end{array}}$$

$$= \left[48t - 16t^2\right]_0^{3/2} + \left[16t^2 - 48t\right]_{3/2}^{2} = 40.$$