MATH 132H FALL 2012 FINAL EXAM

Your Name: My Solution

This is a two hours exam. This exam paper consists of 7 questions. It has 9 pages.

On this exam, you may use a calculator and one letter size page of notes, but no books.

It is not sufficient to just write the answers. You must *explain* how you arrive at your answers.

1. (14)	
2. (14)	
3. (14)	
4. (14)	
5. (14)	
6. (18)	
7. (18)	
TOTAL (106)	

1. (14 points) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(5n+1)}$. Justify your answer (do not forget to justify the convergence or divergence at the endpoints).

Radius of convergence: 7 pts

n-th root test $\lim_{n\to\infty} \left| \frac{(x-2)^n}{3^n(5n+1)} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x-2|}{3 \underbrace{(5n+1)^{\frac{1}{n}}}_{\downarrow\, n\to\infty \atop 1}} = \frac{|x-2|}{3} < 1$

$|x-2| < 3$

$-3 < x-2 < 3$

$-1 < x < 5$

Convergence at the endpoints:

When $x = 5$, the series becomes $\sum_{n=0}^{\infty} \frac{1}{5n+1}$, which diverges by the comparison test with $\sum_{n=0}^{\infty} \frac{1}{5(n+1)} = \left(\frac{1}{5}\right) \sum_{n=1}^{\infty} \frac{1}{n}$,

When $x = -1$, the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$.

The sequence $b_n = \frac{1}{5n+1}$ is decreasing, $\lim_{n\to\infty} \frac{1}{5n+1} = 0$, and so the alternating series is convergent, by the Alternating sequence Theorem. (It is conditionally convergent, since $\sum_{n=0}^{\infty} \frac{1}{5n+1}$ is divergent).

Interval of convergence: $[-1, 5)$.

$\underset{\sim}{7}$ pts

2. (14 points) a) Find the Maclaurin series for $f(x) = \dfrac{1}{1+x^2}$. Justify your answer!

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n \quad , \text{ for } |u|<1. \quad \text{Hence}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$\underset{\sim}{7}$ pts

b) Use your answer in part (a) in order to find the Macluarin series of $\tan^{-1}(x)$. Prove your answer.

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+x^2}\,dx$$ (the anti derivative with value 0 at $x=0$).

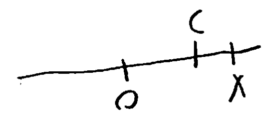
Integrating the Maclaurin series of $\frac{1}{1+x^2}$ term by term we get:

$$\sum_{n=0}^{\infty} \int_0^x (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$|R_m(x)| \le \frac{M}{(m+1)!}|x|^{m+1}, \quad \text{where } |f^{(m+1)}(x)| < M \text{ for all } x \text{ in the interval}$$

3. (14 points) Use Taylor's Inequality and the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ in order to find the minimal n, such that the n-th Taylor polynomial $T_n(x)$ approximates e^x with an error $\le 10^{-5}$ in the interval $-1 \le x \le 1$.

Taylor's inequality



$$\underbrace{f(x) - T_m(x)}_{R_m(x)} = \frac{f^{(m+1)}(c)\, x^{m+1}}{(m+1)!} \quad \text{for } c \text{ between } 0 \text{ and } x,$$

If $x \in [-1,1]$, then $c \in [-1,1]$ and $f^{(m+1)}(c) = e^c < e^1$, since e^x is increasing. So

$$|R_m(x)| \le \frac{e^1 |x|^{m+1}}{(m+1)!} \underset{|x|<1}{\le} \frac{e}{(m+1)!}.$$

Need $\frac{e}{(m+1)!} \le 10^{-5} \iff (m+1)! \ge \underbrace{e \cdot 10^5}_{\vee \; 270000}$

$m=7$ $\quad 8! = 40320$

$\boxed{m=8}$ $\quad 9! = 362\,888 > e \cdot 10^5$

absolutely

4. (14 points) Determine whether each of the following series is convergent, conditionally convergent, or divergent. Explain which test you used and why all the conditions of the test are satisfied.

7 pts

a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$.

Ratio test: $\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{(n+1)!} \Big/ \frac{2^n}{n!} = \frac{2}{n+1} \xrightarrow{n\to\infty} 0 < 1$

So the series is convergent absolulely, by the ratio test

7 pts

b) $\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{\sqrt{n}+\ln(n)}{1+n^2}}_{a_n}$.

Use the limit comparison test

with $\frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{|a_n|}{1/n^{3/2}} = \lim_{n\to\infty} \frac{n^2 + \ln(n) \cdot n^{3/2}}{1+n^2} = \lim_{n\to\infty} \frac{1 + \overbrace{\left(\frac{\ln(n)}{\sqrt{n}}\right)}^{(*)\ 0}}{\frac{1}{n^2} + 1} = \frac{1}{1} = 1,$$

(*) $$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \underset{L'Hop}{\left[\frac{\infty}{\infty}\right]} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, by the p-test $p = 3/2 > 1$,

Thus $\sum |a_n|$ converges and $\sum a_n$ converges absolutely.

7 pts

5. (14 points) Compute the following integrals **algebraically**. Show all your work!

a) $\int_1^\infty \frac{\overset{2du}{dx}}{\sqrt{x}(\sqrt{x}+1)(\ln(\underbrace{\sqrt{x}+1}_{u}))^2} = \int_{u=2}^{\infty} \frac{2\,du}{u(\ln(u))^2} =$

$du = \frac{1}{2\sqrt{x}}\,dx$

$t = \ln(u)$

$dt = \frac{1}{u}\,du$

$$= \int_{\ln(2)}^{\infty} \frac{2\,dt}{t^2} = \lim_{T\to\infty}\left[-\frac{2}{t}\right]_{\ln(2)}^{T} = \lim_{T\to\infty}\left(-\frac{2}{T} + \frac{2}{\ln(2)}\right) = \boxed{\frac{2}{\ln(2)}}$$

7 pts

b) $I = \int \underset{u}{\cos(x)}\underset{v'}{e^{2x}}\,dx = \frac{1}{2}\cos(x)e^{2x} + \frac{1}{2}\int \underset{u}{\sin(x)}\,\underset{v'}{e^{2x}}\,dx$

$u' = -\sin(x) \quad v = \frac{1}{2}e^{2x}$

$u' = \cos(x) \quad v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}\cos(x)e^{2x} + \frac{1}{2}\left[\frac{1}{2}\sin(x)e^{2x} - \frac{1}{2}\underbrace{\int \cos(x)e^{2x}\,dx}_{I}\right]$$

$$= \frac{1}{2}\cos(x)e^{2x} + \frac{1}{4}\sin(x)e^{2x} - \frac{1}{4}I + C$$

$$\frac{5}{4}I = \frac{1}{2}\cos(x)e^{2x} + \frac{1}{4}\sin(x)e^{2x} + C$$

$$\boxed{I = \frac{2}{5}\cos(x)e^{2x} + \frac{1}{5}\sin(x)e^{2x} + C'}$$

6 pt

6. (18 points) Consider the curve given by the parametric equations $x = e^{-t}\cos(t)$, $y = e^{-t}\sin(t)$, $0 \le t \le 2\pi$.

a) Find all the point where the tangent line to the curve is horizontal and all the point where the tangent line is vertical. Show all your algebraic steps.

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-e^{-t}\sin(t) + e^{-t}\cos(t)}{-e^{-t}\cos(t) - e^{-t}\sin(t)} = -\left(\frac{-\sin(t)+\cos(t)}{\cos(t)+\sin(t)}\right)$$

Horizontal Tangent line: $\sin(t) = \cos(t)$, when $t = \frac{\pi}{4} + 2\pi$, so

$t_1 = \frac{\pi}{4}$, $t_2 = \frac{5\pi}{4}$.

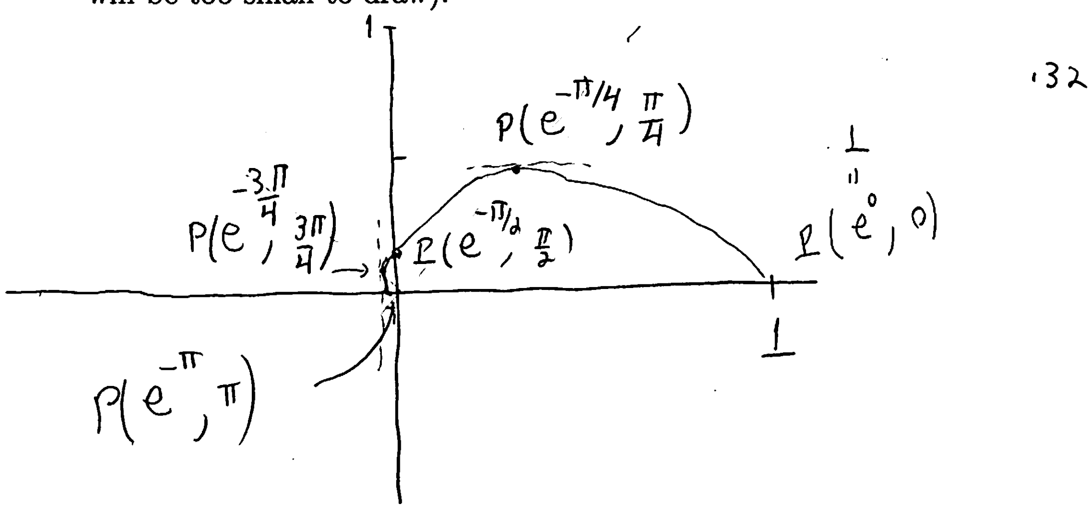
$e^{-\pi/4}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad e^{-5\pi/4}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Vertical Tangent line: $\cos(t) = -\sin(t)$, $t = \frac{3\pi}{4}, \frac{7\pi}{4}$.

$e^{-\frac{3\pi}{4}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\quad e^{-\frac{7\pi}{4}}\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

6 pts

b) Sketch the graph of the portion of the curve for $0 \le t \le \pi$. Indicate the polar coordinates of the points of intercept with the x and y axis, the scale, and the horizontal and vertical tangent lines and the polar coordinates of the points with these tangent lines. (The portion of the curve in the interval $\pi < t \le 2\pi$ will be too small to draw).



t	x	y
0	1	0
$\frac{\pi}{4}$	$\frac{e^{-\pi/4}}{\sqrt{2}}$	$\frac{e^{-\pi/4}}{\sqrt{2}}$
$\frac{\pi}{2}$	0	$e^{-\frac{\pi}{2}} = 0.20$
$\frac{3\pi}{4}$		
π	$-e^{-\pi} = -0.04$	0

6 pts

c) Find the length of the curve in part (a).

$$L = \int_0^{2\pi} \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\, dt = \int_0^{2\pi} \sqrt{e^{-2t}\underbrace{(\cos(t) - \sin(t))^2}_{\cos^2(t) - \cancel{2\cos(t)\sin(t)} + \sin^2(t)} + e^{-2t}\underbrace{((\cos(t) + \sin(t))^2}_{\cos^2(t) + \cancel{2\cos(t)\sin(t)} + \sin^2(t)})}\, dt$$

$$= \int_0^{2\pi} e^{-t} \cdot \sqrt{2}\, dt = \sqrt{2}\left[-e^{-t}\right]_0^{2\pi} =$$

$$= \sqrt{2}\left(1 - e^{-2\pi}\right)$$

6 pt

7. (18 points) a) Find the cartesian equation of the polar curve $r = 5\sin(\theta)$.

$$\left.\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\} + r = 5\sin(\theta)$$

$$5y = r \cdot \underbrace{5\sin(\theta)}_{r} = r^2 = x^2 + y^2$$

$$\boxed{y^2 - 5y + x^2 = 0}$$

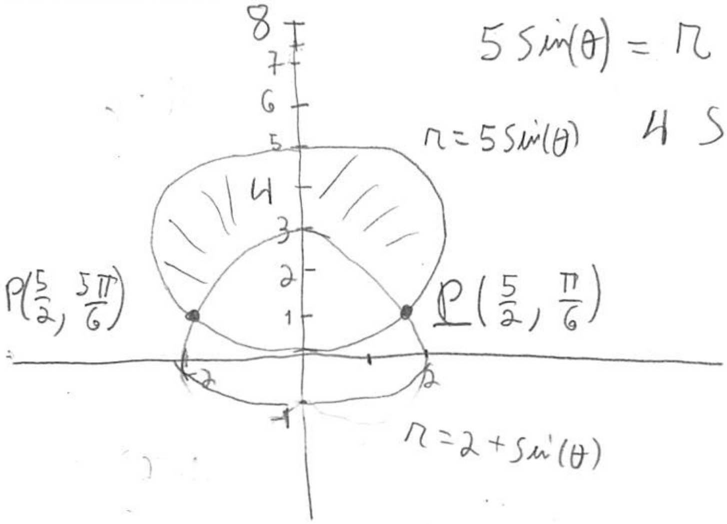
$$\boxed{\left(y - \frac{5}{2}\right)^2 + x^2 = \left(\frac{5}{2}\right)^2}$$

circle of radius $\frac{5}{2}$ centered at $\left(0, \frac{5}{2}\right)$.

6 pts

b) Sketch the region that lies inside the polar curve $r = 5\sin(\theta)$, from part (a), and outside the polar curve $r = 2 + \sin(\theta)$. Provide polar coordinates for all points of intersection.

θ	r
$-\pi/2$	1
0	2
$\pi/2$	3



$5\sin(\theta) = r = 2 + \sin(\theta)$

$4\sin(\theta) = 2$

$\sin(\theta) = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

6 pts

c) Find the area of the region in part (b).

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} \underbrace{[5\sin(\theta)]^2}_{25\sin^2(\theta)} - \underbrace{[2+\sin(\theta)]^2}_{4+4\sin(\theta)+\sin^2(\theta)} \, d\theta = \frac{1}{\cancel{2}}\int_{\pi/6}^{5\pi/6} \underbrace{\cancel{24}\sin^2(\theta)}_{12} - \underbrace{\cancel{4}\sin(\theta)}_{2} - \underbrace{\cancel{4}}_{2}\, d\theta$$

$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

$$6 - 6\cos(2\theta)$$

$$= \int_{\pi/6}^{5\pi/6} 4 - 6\cos(2\theta) - 2\sin(\theta)\, d\theta =$$

$$= \Big[4\theta + 3\sin(2\theta) + 2\cos(\theta)\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{8\pi}{3} + 2\Big[\underbrace{\cos(5\pi/6) - \cos(\tfrac{\pi}{6})}_{-2\frac{\sqrt{3}}{2} = -\sqrt{3}}\Big] = \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$