Math 132, Fall 2009
Exam 2: Solutions

(1) a) (12 points) Determine for which positive real numbers p, is the following improper integral convergent, and for which it is divergent. Evaluate the integral for each value of p, for which it converges, and express its value in terms of p. Justify your answer and show all your algebraic steps. Hint: Do not forget to consider the case $p = 1$ as well.
$\int_e^\infty \frac{1}{(\ln(x))^p x}\, dx$

Let $u = \ln(x)$ so $du = 1/x\, dx$. Then

$$\begin{aligned}\int_e^\infty \frac{1}{(\ln(x))^p x}\, dx &= \lim_{N\to\infty} \int_e^N \frac{1}{(\ln(x))^p x}\, dx \\ &= \lim_{N\to\infty} \int_1^{\ln(N)} \frac{1}{u^p}\, du \\ &= \lim_{N\to\infty} \int_1^N \frac{1}{u^p}\, du\end{aligned}$$

Case 1: $p = 1$.

$$\begin{aligned}\int_e^\infty \frac{1}{(\ln(x))^p x}\, dx &= \lim_{N\to\infty} \int_1^N \frac{1}{u}\, du \\ &= \lim_{N\to\infty} [\ln|u|]_1^N \\ &= \lim_{N\to\infty} (\ln(N) - \ln(1)) \\ &= \infty\end{aligned}$$

So, for $p = 1$, the integral is divergent.

Case 2: $p \neq 1$.

$$\begin{aligned}\int_e^\infty \frac{1}{(\ln(x))^p x}\,dx &= \lim_{N\to\infty}\int_1^N \frac{1}{u^p}\,du\\ &= \lim_{N\to\infty} \left.\frac{u^{1-p}}{1-p}\right|_1^N\\ &= \frac{1}{1-p}\lim_{N\to\infty}\left(N^{1-p}-1\right)\end{aligned}$$

Now

$$\lim_{N\to\infty} N^{1-p} = \begin{cases} 0 & \text{if } p>1\\ \infty & \text{if } p<1\end{cases}$$

so

$$\frac{1}{1-p}\lim_{N\to\infty}\left(N^{1-p}-1\right) = \begin{cases} \frac{1}{p-1} & \text{if } p>1\\ \infty & \text{if } p<1\end{cases}$$

Then the integral is divergent for $p<1$ and convergent for $p>1$.

We have

$$\int_e^\infty \frac{1}{(\ln(x))^p x}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1\\ \text{divergent} & \text{if } p\leq 1\end{cases}$$

b) (10 points) Determine whether the following improper integral converges or diverges. Evaluate it, showing all your algebraic steps, if it is convergent. Otherwise, explain why it is divergent.

$\int_0^8 \frac{1}{(x-8)^{(2/3)}}\,dx$

$$\begin{aligned}\int_0^8 \frac{1}{(x-8)^{2/3}}\,dx &= \lim_{N\to 8^-}\int_0^N \frac{1}{(x-8)^{2/3}}\,dx\\ &= \lim_{N\to 8^-} 3(x-8)^{1/3}\Big|_0^N\\ &= \lim_{N\to 8^-}\left(3(N-8)^{1/3}-3(-8)^{1/3}\right)\\ &= 6\end{aligned}$$

(2) (14) For each of the following **sequences** (not series) determine whether the sequence converges or diverges. If it converges, find the limit, showing all your algebraic steps. Otherwise, explain why it diverges.

a) $a_n = \dfrac{\sqrt{2n^2+3}}{3n-1}, \quad n \geq 1.$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sqrt{2n^2+3}}{3n-1} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}\sqrt{2n^2+3}}{\frac{1}{n}(3n-1)} \\
&= \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n^2}(2n^2+3)}}{3-\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2+\frac{3}{n^2}}}{3-\frac{1}{n}} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}$$

Then the sequence converges to $\sqrt{2}/3$.

b) $a_n = n\sin(n)e^{-n}, \quad n \geq 1.$

Since $-1 \leq \sin(n) \leq 1$ and $ne^{-n} \geq 0$,

$$-ne^{-n} \leq n\sin(n)e^{-n} \leq ne^{-n}$$

Now,

$$\begin{aligned}
\lim_{n\to\infty} ne^{-n} &= \lim_{x\to\infty} xe^{-x} \\
&= \lim_{x\to\infty} \frac{x}{e^x} \to \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{1}{e^x} \qquad \text{by L'Hospital's rule} \\
&= 0
\end{aligned}$$

Also,

$$\lim_{n\to\infty} -ne^{-n} = -\lim_{n\to\infty} ne^{-n} = 0.$$

By the Squeeze Theorem,

$$\lim_{n\to\infty} n\sin(n)e^{-n} = 0.$$

Therefore, the sequence converges to 0.

(3) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

(a) (9 points) Use the integral test to show that the series is convergent. Show that all the hypothesis of the test are satisfied. Show all your algebraic steps. (Credit will not be given for an answer using another test).

Let $f(x) = \frac{1}{x^6}$ so that $f(n) = \frac{1}{n^6}$ for each integer $n \geq 1$.

1. $f(x) = 1/x^6$ is continuous for $x \geq 1$.
2. $f(x) = 1/x^6 \geq 0$ for $x \geq 1$.
3. $f'(x) = -6x^{-7} < 0$ for $x \geq 1$. So, f is decreasing for $x \geq 1$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^6}\,dx &= \lim_{N\to\infty} \int_1^N \frac{1}{x^6}\,dx \\
&= \lim_{N\to\infty} -\frac{1}{5}x^{-5}\Big|_1^N \\
&= \lim_{N\to\infty} -\frac{1}{5}(N^{-5} - 1) \\
&= \frac{1}{5}
\end{aligned}$$

Since the integral $\displaystyle\int_1^{\infty} \frac{1}{x^6}\,dx$ is convergent, by the integral test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$ is also convergent.

(b) (9 points) Let s be the sum of the series in part 3a. Find the **minimal** number n of terms of the series, for which we know that $s - s_n \leq 0.00001$, by the error estimate of the integral test. Justify your answer, showing all your algebraic steps.

From part a, the function $f(x) = 1/x^6$ is continuous, positive and decreasing for $x \geq 1$. Since $f(n) = 1/n^6$, by the error estimate for the integral test,

$$s - s_n \leq \int_n^\infty \frac{1}{x^6}\, dx$$

Now

$$\begin{aligned}
\int_n^\infty \frac{1}{x^6}\, dx &= \lim_{N\to\infty} \int_n^N \frac{1}{x^6}\, dx \\
&= \lim_{N\to\infty} -\frac{1}{5}x^{-5}\bigg|_n^N \\
&= \lim_{N\to\infty} -\frac{1}{5}(N^{-5} - n^{-5}) \\
&= \frac{1}{5}n^{-5}
\end{aligned}$$

So,

$$s - s_n \leq \frac{1}{5}n^{-5}$$

Since we want $s - s_n \leq 0.00001$,

$$s - s_n \leq \frac{1}{5}n^{-5} \leq 0.00001$$

or

$$\begin{aligned}
\frac{1}{5}n^{-5} &\leq 0.00001 \\
n^{-5} &\leq 0.00005 \\
n^5 &\geq 20000 \\
n^5 &\geq 20000^{1/5} \approx 7.25
\end{aligned}$$

Therefore, we need at least $n = 8$ terms to conclude $s - s_n \leq 0.00001$ from the error estimate for the integral test.

(4) (14 points) Consider the series $\sum_{n=1}^{\infty} \frac{5}{3+2^n}$

(a) (5 points) Use the comparison test to show that the series converges.

For each $n \geq 1$, $\frac{5}{3+2^n} \geq 0$. And

$$2^n \leq 3 + 2^n \qquad \Rightarrow \qquad \frac{1}{3+2^n} \leq \frac{1}{2^n} \qquad \Rightarrow \qquad \frac{5}{3+2^n} \leq \frac{5}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{5}{2^n} = \sum_{n=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^{n-1}$ is a geometric series with $r = 1/2 < 1$, so it is convergent. Therefore, by the comparison test, the series $\sum_{n=1}^{\infty} \frac{5}{3+2^n}$ is also convergent.

(b) (9 points) The sum s_{10} of the first 10 terms of the series, rounded to five decimal digits, is 2.72152. You do not need to verify this. Show that $s - 2.72152$ is less than 0.01. Justify your answer!

Solution 1.

For a convergent geometric series

$$t = \sum_{n=1}^{\infty} ar^{n-1}, \qquad r < 1,$$

the partial sums are

$$t_n = \frac{a(1-r^n)}{1-r}$$

and the sum is

$$t = \frac{a}{1-r}.$$

Let $a = 5/2$ and $r = 1/2$ so that

$$\begin{aligned} t &= \sum_{n=1}^{\infty} \frac{5}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} \\ &= \frac{5/2}{1-1/2} \\ &= 5 \end{aligned}$$

and

$$t_n = \frac{(5/2)(1-(1/2)^n)}{1-(1/2)} = 5(1-(1/2)^n)$$

Since $\frac{5}{3+2^n} \leq \frac{5}{2^n}$ for all n, we have

$$s - s_n \leq t - t_n$$

for all n. Then for $n = 10$,

$$\begin{aligned} s - s_{10} &\leq t - t_{10} \\ &= 5 - 5(1-(1/2)^{10}) \\ &= 5/2^{10} \\ &< 0.01 \end{aligned}$$

Solution 2.

Let $t = \displaystyle\sum_{n=1}^{\infty} \frac{5}{2^n}$, t_n be the partial sums and $f(x) = 5/2^x$.

1. $f(x) = 5/2^x$ is continuous for all x.
2. $f(x) = 5/2^x > 0$ for all x.
3. $f(x) = 5(2^{-x})$ so $f'(x) = -5(\ln 2)2^{-x} < 0$. Then f is decreasing for all x.

Therefore, by the error estimate for the integral test,

$$t - t_n \leq \int_n^{\infty} \frac{5}{2^x}\, dx$$

Since $\frac{5}{3+2^n} \leq \frac{5}{2^n}$ for all n, we also have

$$s - s_n \leq t - t_n$$

for all n. So for $n = 10$,

$$\begin{aligned}
s - s_{10} &\leq t - t_{10} \\
&\leq \int_{10}^{\infty} \frac{5}{2^x}\, dx \\
&= \lim_{N\to\infty} \int_{10}^{N} \frac{5}{2^x}\, dx \\
&= \lim_{N\to\infty} \int_{10}^{N} 5(2^{-x})\, dx \\
&= \lim_{N\to\infty} \frac{-5}{\ln 2} 2^{-x}\Big|_{10}^{N} \\
&= \lim_{N\to\infty} \frac{-5}{\ln 2} \left(2^{-N} - 2^{-10}\right) \\
&= \frac{5(2^{-10})}{\ln 2} \\
&< 0.01
\end{aligned}$$

(5) (32 points) Determine whether the following series converge absolutely, converge conditionally, or diverge. Name each test you use and indicate why all the conditions needed for it to apply actually hold.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{5n^3-2n^2+1}$

Let $a_n = \frac{2n+5}{5n^3-2n^2+1}$ and $b_n = \frac{2}{5n^2}$. Then $a_n, b_n > 0$ and

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{2n+5}{5n^3-2n^2+1}}{\frac{2}{5n^2}} \\
&= \lim_{n\to\infty} \frac{2n+5}{5n^3-2n^2+1}\left(\frac{5n^2}{2}\right) \\
&= \lim_{n\to\infty} \frac{10n^3+25n^2}{10n^3-4n^2+2} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n^3}(10n^3+25n^2)}{\frac{1}{n^3}(10n^3-4n^2+2)} \\
&= \lim_{n\to\infty} \frac{10+25/n}{10-4/n+2/n^3)} \\
&= 1.
\end{aligned}$$

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ is positive and finite, by the limit comparison test either both series are convergent or both series are divergent. The series $\displaystyle\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ so it is convergent. Since $\displaystyle\sum \frac{2}{5n^2} = \frac{2}{5}\sum \frac{1}{n^2}$, the series $\displaystyle\sum \frac{2}{5n^2}$ is also convergent. Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{5n^3-2n^2+1}$ is convergent. Since the terms are positive, the series is absolutely convergent.

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{4^n}{n!}\right)$

Let $a_n = (-1)^{n-1}\left(\frac{4^n}{n!}\right)$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^n\left(\frac{4^{n+1}}{(n+1)!}\right)}{(-1)^{n-1}\left(\frac{4^n}{n!}\right)}\right| \\
&= \lim_{n\to\infty}\left(\frac{4^{n+1}}{(n+1)!}\right)\left(\frac{n!}{4^n}\right) \\
&= \lim_{n\to\infty}\frac{4}{n+1} \\
&= 0 \\
&< 1.
\end{aligned}$$

Thus, the series is absolutely convergent by the ratio test.

(c) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{2n^2+n}{3n^2+7n}\right)^{2n}$

Let $a_n = (-1)^n\left(\frac{2n^2+n}{3n^2+7n}\right)^{2n}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \left(\frac{2n^2+n}{3n^2+7n}\right)^{2n}\right|} \\
&= \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n^2+n}{3n^2+7n}\right)^{2n}} \\
&= \lim_{n\to\infty} \left(\frac{2n^2+n}{3n^2+7n}\right)^2 \\
&= \left(\lim_{n\to\infty} \frac{2n^2+n}{3n^2+7n}\right)^2 \\
&= \left(\lim_{n\to\infty} \frac{(2n^2+n)/n^2}{(3n^2+7n)/n^2}\right)^2 \\
&= \left(\lim_{n\to\infty} \frac{2+1/n}{3+7/n}\right)^2 \\
&= (2/3)^2 \\
&= 4/9 \\
&< 1.
\end{aligned}$$

Thus, the series is absolutely convergent by the root test.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(\frac{n+2}{n^2+4}\right)$

This is an alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^n b_n$ with $b_n = \frac{n+2}{n^2+4}$.

1.

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n+2}{n^2+4} \\
&= \lim_{n\to\infty} \frac{1+2/n}{n+4/n} \\
&= 0.
\end{aligned}$$

2. Let $f(x) = \frac{x+2}{x^2+4}$ so that $f(n) = b_n$.

$$\begin{aligned} f'(x) &= \frac{(x^2+4) - 2x(x+2)}{(x^2+4)^2} \\ &= \frac{-x^2 - 4x + 4}{(x^2+4)^2} \end{aligned}$$

$-x^2 - 4x \leq -4x \leq -4$ for $x \geq 1$. So, $-x^2 - 4x + 4 \leq 0$ for $x \geq 1$. Then f is decreasing for $x \geq 1$ which implies $b_{n+1} \leq b_n$ for $n \geq 1$. Hence, by the alternating series test, the series is convergent.

Let $a_n = (-1)^n \left(\frac{n+2}{n^2+4}\right)$ so that $|a_n| = \frac{n+2}{n^2+4} > 0$. Let $c_n = \frac{1}{n} > 0$. Then

$$\begin{aligned} \lim_{n\to\infty} \frac{|a_n|}{c_n} &= \lim_{n\to\infty} \frac{\frac{n+2}{n^2+4}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n^2+2n}{n^2+4} \\ &= \lim_{n\to\infty} \frac{1+2/n}{1+4/n^2} \\ &= 1 \end{aligned}$$

Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ is positive and finite, by the limit comparison test either both series are convergent or both series are divergent. The series $\displaystyle\sum \frac{1}{n}$ is a p-series with $p = 1$ so it is divergent. Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^2+4}$ is also divergent.

Therefore, the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+2}{n^2+4}\right)$ is conditionally convergent.