(1) (21 points) Evaluate the following integrals using substitution. Explicitly show any relevant algebraic manipulation.

a) \( \int \sin^2(x) \cos^3(x) \, dx \)

Let \( u = \sin(x) \) so \( du = \cos(x) \, dx \). Then

\[
\int \sin^2(x) \cos^3(x) \, dx = \int u^2(1 - u^2) \, du \\
= \int (u^2 - u^4) \, du \\
= \frac{u^3}{3} - \frac{u^5}{5} + C \\
= \frac{1}{3} \sin^3(x) - \frac{1}{5} \sin^5(x) + C
\]

b) \( \int_1^2 \frac{e^{2/x}}{x^2} \, dx \)

Let \( u = \frac{2}{x} \) so \( du = -\frac{2}{x^2} \, dx \).

\[
x = 1 \quad \Rightarrow \quad u = 2/1 = 2 \\
x = 2 \quad \Rightarrow \quad u = 2/2 = 1
\]
Then

\[ \int_1^2 \frac{e^{2/x}}{x^2} \, dx = -\frac{1}{2} \int_2^1 e^u \, du \]
\[ = -\frac{1}{2} e^u \bigg|_2^1 \]
\[ = -\frac{1}{2} (e - e^2) \]
\[ = -\frac{1}{2} (e - e^2) \]

(c) \[ \int_0^a x \sqrt{a^2 - x^2} \, dx, \text{ where } a > 0. \]

Let \( u = a^2 - x^2 \) so \( du = -2x \, dx. \)

\[ x = 0 \quad \Rightarrow \quad u = a^2 - 0^2 = a^2 \]

\[ x = a \quad \Rightarrow \quad u = a^2 - a^2 = 0 \]

Then

\[ \int_0^a x \sqrt{a^2 - x^2} \, dx = -\frac{1}{2} \int_{a^2}^0 \sqrt{u} \, du \]
\[ = -\frac{1}{2} \left( \frac{2}{3} \right) u^{3/2} \bigg|_0^{a^2} \]
\[ = -\frac{1}{3} (0^{3/2} - (a^2)^{3/2}) \]
\[ = \frac{1}{3} a^3 \]
(2) (18 points) Evaluate the following integrals showing all your algebraic steps. Give the exact value and not a numerical approximation.

a) (8 points)
\[
\int_{0}^{\pi/4} \cos^2(2x)\,dx = \int_{0}^{\pi/4} \frac{1 + \cos(4x)}{2} \,dx \\
= \frac{1}{2} \int_{0}^{\pi/4} 1 + \cos(4x) \,dx \\
= \frac{1}{2} \left[ x + \frac{1}{4} \sin(4x) \right]_{0}^{\pi/4} \\
= \frac{1}{2} \left( \frac{\pi}{4} + \frac{1}{4} \sin(\pi) \right) - \frac{1}{2} \left( 0 + \frac{1}{4} \sin(0) \right) \\
= \frac{\pi}{8}
\]

b) (10 points) \( \int_{0}^{1} t^2 e^{2t} \,dt \).

Integration by parts: \( \int_{a}^{b} u \,dv = uv\big|_{a}^{b} - \int_{a}^{b} v \,du \).

Let
\[
u = t^2 \quad dv = e^{2t} \,dt \\
du = 2t \,dt \quad v = \frac{1}{2} e^{2t}
\]

Then
\[
\int_{0}^{1} t^2 e^{2t} \,dt = \frac{1}{2} \left. t^2 e^{2t} \right|_{0}^{1} - \int_{0}^{1} \frac{1}{2} e^{2t} (2t) \,dt \\
= \frac{e^2}{2} - \int_{0}^{1} t e^{2t} \,dt \\
\]
Integrate by parts again: \( \int_{a}^{b} \tilde{u} \,\tilde{v} \,d\tilde{u} = \tilde{u}\tilde{v}\big|_{a}^{b} - \int_{a}^{b} \tilde{v} \,d\tilde{u} \).

Let
\[
\tilde{u} = t \quad d\tilde{v} = e^{2t} \,dt \\
d\tilde{u} = dt \quad \tilde{v} = \frac{1}{2} e^{2t}
\]
$$\int_0^1 t^2 e^{2t} dt = \frac{e^2}{2} - \int_0^1 te^{2t} dt$$

$$= \frac{e^2}{2} - \left( \frac{1}{2}te^{2t}\bigg|_0^1 - \int_0^1 \frac{1}{2}e^{2t} dt \right)$$

$$= \frac{e^2}{2} - \frac{e^2}{2} + \frac{1}{2}\int_0^1 e^{2t} dt$$

$$= \frac{1}{4}e^{2t}\bigg|_0^1$$

$$= \frac{1}{4}(e^2 - 1)$$

(3) (17) Let $R$ be the region bounded by the $x$-axis, the vertical line $x = 1$, and the graph of $y = x^3$. The region is rotated about the horizontal line $y = -1$ generating a solid. Set up a definite integral for the volume of the solid. Do NOT evaluate the integral.

Figure 1. Region R
Vertical cross-sections of the solid are circular annuli with inner radii 1 and outer radii $1 + x^3$. The area of a cross-section is $A(x) = \pi (1 + x^3)^2 - \pi$ and the volume is

$$V = \int_{0}^{1} \pi ((1 + x^3)^2 - 1) \, dx$$

(4) (a) (5 points) Find the points at which the curves $y = x + 2$ and $y = 3\sqrt[3]{x}$ (three times square root of $x$) intersect. Then sketch the two curves and indicate the region enclosed between them.

\[
\begin{align*}
  x + 2 &= 3\sqrt{x} \\
  (x + 2)^2 &= (3\sqrt{x})^2 \\
  x^2 + 4x + 4 &= 9x \\
  x^2 - 5x + 4 &= 0 \\
  (x - 1)(x - 4) &= 0 \\
  x &= 1 \quad \text{or} \quad x = 4
\end{align*}
\]

So, the curves intersect at the points $(1, 3)$ and $(4, 6)$.

**Figure 2**
(b) (11 points) Calculate the area of that region enclosed between the two curves. Show all your algebraic steps.

\[ A = \int_1^4 3\sqrt{x} - (x + 2) \, dx \]
\[ = 2x^{3/2} - \frac{x^2}{2} - 2x \bigg|_1^4 \]
\[ = 2 \left( 4^{3/2} \right) - \frac{4^2}{2} - 2(4) - \left( 2 - \frac{1}{2} - 2 \right) \]
\[ = 16 - 8 - 8 - \frac{1}{2} + 2 \]
\[ = \frac{1}{2} \]

(5) (a) (5 points) Set \( F(x) := \int_0^x e^{(t^2)} \, dt \). Calculate the derivative \( F'(x) \). Justify your answer.

Since \( e^{t^2} \) is continuous for all \( t \), by the Fundamental Theorem of Calculus Part I,

\[ F'(x) = e^{x^2} \]

(b) (5 points) Set \( G(x) := \int_0^{3x^2} e^{(t^2)} \, dt \). Calculate the derivative \( G'(x) \). Justify your answer and show all your algebraic steps.

Let \( u = 3x^2 \) so \( \frac{du}{dx} = 6x \).
\[ F'(x) = \frac{d}{dx} \int_{0}^{3x^2} e^{t^2} \, dt \]

\[ = \frac{d}{dx} \int_{0}^{u} e^{t^2} \, dt \]

\[ = \frac{d}{du} \left( \int_{0}^{u} e^{t^2} \, dt \right) \frac{du}{dx} \quad \text{by the Chain Rule} \]

\[ = e^{x^2} (6x) \quad \text{by the Fundamental Theorem of Calculus Part I} \]

\[ = 6xe^{x^2} \]

(6) (18 points) A particle moves along the \( x \)-axis in a straight line with velocity \( v(t) = t^2 - 5t + 6 \) for \( 0 \leq t \leq 3 \) (measured in ft/sec).

a) Determine the \textbf{total displacement} of the particle between the times \( t = 0 \) and \( t = 3 \).

Displacement = \( \int_{0}^{3} v(t) \, dt \)

\[ = \int_{0}^{3} (t^2 - 5t + 6) \, dt \]

\[ = \left. \frac{t^3}{3} - \frac{5t^2}{2} + 6t \right|_{0}^{3} \]

\[ = \frac{3^3}{3} - \frac{5(3^2)}{2} + 6(3) \]

\[ = 9 - \frac{45}{2} + 18 \]

\[ = 27 - \frac{45}{2} \]

\[ = \frac{54}{2} - \frac{45}{2} \]

\[ = \frac{9}{2} \]
b) Determine the **total distance** that the particle traveled between \( t = 0 \) and \( t = 3 \).

\[
\text{Distance} = \int_{0}^{3} |v(t)| \, dt
\]

First we find the zeroes of \( v(t) \).

\[
v(t) = t^2 - 5t + 6 = 0
\]

\[ (t - 2)(t - 3) = 0 \]

\[ t = 2 \quad \text{or} \quad t = 3 \]

Between \( t = 0 \) and \( t = 2 \), \( v(t) \geq 0 \) and between \( t = 2 \) and \( t = 3 \), \( v(t) \leq 0 \). Then for \( 0 \leq t \leq 3 \),

\[
|v(t)| = \begin{cases} 
 v(t) & \text{if } 0 \leq t \leq 2 \\
 -v(t) & \text{if } 2 \leq t \leq 3 
\end{cases}
\]

\[ \text{Figure 3. } v(t) = t^2 - 5t + 6 \]
Distance = $\int_{0}^{3} |v(t)| \, dt$

$= \int_{0}^{2} v(t) \, dt + \int_{2}^{3} -v(t) \, dt$

$= \int_{0}^{2} (t^2 - 5t + 6) \, dt + \int_{2}^{3} -(t^2 - 5t + 6) \, dt$

$= \frac{t^3}{3} - \frac{5t^2}{2} + 6t \bigg|_{0}^{2} - \frac{t^3}{3} + \frac{5t^2}{2} - 6t \bigg|_{2}^{3}$

$= \frac{2^3}{3} - \frac{5(2^2)}{2} + 6(2) - \frac{3^3}{3} + \frac{5(3^2)}{2} - 6(3) - \left( -\frac{2^3}{3} + \frac{5(2^2)}{2} - 6(2) \right)$

$= \frac{8}{3} - 10 + 12 - 9 + \frac{45}{2} - 18 + \frac{8}{3} - 10 + 12$

$= -23 + \frac{45}{2} + \frac{16}{3}$

$= \frac{138}{6} + \frac{135}{6} + \frac{32}{6}$

$= \frac{29}{6}$