

1 Introduction

Let $R := \mathbb{C}[x]$ be the ring of polynomials. Let $V_{n,d}$ be the vector space of all $n \times n$ matrices with entries in $R$, such that the degree of each entry is $\leq d$. Clearly, $\dim(V_{n,d}) = n^2(d+1)$. Given a matrix $A = (a_{ij}(x))$ in $V_{n,d}$, its characteristic polynomial

$$\text{char}_A(x, \lambda) := \det[A - \lambda I]$$

is a polynomial in two variables. The zero locus of $\text{char}_A(x, \lambda)$ is an affine plane curve, called the affine spectral curve of $A$. Algebraic curves very often arise in other branches of mathematics as spectral curves (see [B2] for examples arising in classical mechanics).

In problem 3 below you will prove the following statement, for all $d \geq 1$ and $n \geq 1$. Set $\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1})$ and let $p : \mathbb{F}_d \rightarrow \mathbb{P}^1$ be the natural morphism. Set $M := \mathcal{O}_{\mathbb{F}_d}(1) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(d)$. Let $M^n$ be the $n$-th tensor power of $M$.

**Theorem 1** There exists a Zariski dense open subset of $V_{n,d}$, consisting of matrices $A$, whose affine spectral curve is a Zariski open subset of a smooth connected projective curve $\tilde{C}$ of genus $d \left(\frac{n(n-1)}{2}\right) - n + 1$. The curve $\tilde{C}$ is naturally embedded\(^1\) in the ruled surface $\mathbb{F}_d$ as a divisor in the linear system $|M^n|$.

The construction introduces a morphism $\text{char} : V_{n,d} \rightarrow |M^n|$. In Problem 4 you will describe the fiber $\text{char}^{-1}(\tilde{C})$ in terms of the spectral curve $\tilde{C}$.

Set $F := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}$. Key to the proof is the observation that an element $A$ of $V_{n,d}$ corresponds to a homomorphism of $\mathcal{O}_{\mathbb{P}^1}$-modules $\varphi : F \rightarrow F \otimes \mathcal{O}_{\mathbb{P}^1}(d)$ as follows. Choose homogeneous coordinates $(t_0, t_1)$ over $\mathbb{P}^1$. Set $\varphi_{ij}(t_0, t_1) := t_0^d a_{ij}(t_1/t_0)$. Then $\varphi_{ij}$ is a homogeneous polynomial of degree $d$, hence a section of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. We get the isomorphism

$$V_{n,d} \cong \text{Hom}(F, F \otimes \mathcal{O}_{\mathbb{P}^1}(d)),
(a_{ij}) \mapsto (\varphi_{ij}).$$

N. Hitchin discovered in the 1980’s that spectral curves play an important role in the study of $n$-dimensional irreducible complex representations of the fundamental group of a complex projective curve $C$ of positive genus [H]. Hitchin’s pairs $(F, \varphi)$ consist of a rank $n$ vector bundle $F$ on $C$ and its “endomorphism” $\varphi : F \rightarrow F \otimes \omega_C$ is twisted by the canonical line-bundle $\omega_C$. Hitchin’s spectral curves are embedded in the ruled surface $\mathbb{P}[\omega_C \otimes \mathcal{O}_C]$. The genus of Hitchin’s spectral curve, which you will calculate below, is equal to half the dimension of the space of representations of the fundamental group.

Terminology: A rank $n$ vector bundle over an algebraic variety $X$ is a locally free $\mathcal{O}_X$-module of rank $n$. The following three objects are one and the same: a line-bundle, an invertible sheaf, and a locally free $\mathcal{O}_X$-module of rank 1.

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\(^1\)Note that the closure of such a curve in $\mathbb{P}^2$ has degree $nd$, so arithmetic genus $(nd - 1)(nd - 2)/2$. The latter is larger than the geometric genus by $n(d - 1)[nd - 2]/2$. Hence the closure in $\mathbb{P}^2$ is singular, except possibly when $d = 1$, or $(n, d) = (1, 2)$.
2 Problems

1. Let $C$ be a smooth curve, $L$ a line bundle on $C$ of degree $d$, $E := L \oplus \mathcal{O}_C$, and $p : \mathbb{P}E \to C$ the corresponding ruled surface. The line sub-bundle $L$ of $E$ corresponds to a section $\sigma_\infty : C \to \mathbb{P}E$, whose image is $\Sigma_\infty := \mathbb{P}L$. Let $\sigma_0 : C \to \mathbb{P}E$ be the section corresponding to the line sub-bundle $\mathcal{O}_C$ of $E$, and denote its image by $\Sigma_0$. The fiber of $[\mathbb{P}E \setminus \Sigma_\infty]$ over $y \in C$ can be naturally identified with the fiber $\mathcal{T}_y$ of $L$, and $\sigma_0(y)$ is its zero point. Simply associate to $\ell \in \mathcal{T}_y$ the point in $\mathbb{P}E$ corresponding to the line span$_C\{(\ell, 1)\}$ in the fiber of $E$.

(a) Show that $\Sigma_0$ belongs to the linear system $|(p^*L) \otimes \mathcal{O}_{\mathbb{P}E}(1)|$ and $\Sigma_\infty$ belongs to $|\mathcal{O}_{\mathbb{P}E}(1)|$. Hint: Consider the tautological exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*(E) \to Q_{\mathbb{P}E} \to 0.$$  

Show that the section $(0, 1)$ of $p^*E$ maps to a non-zero section of $Q_{\mathbb{P}E}$, which vanishes along $\Sigma_0$ with multiplicity 1. Then repeat your argument for the section $(1, 0)$ of $p^*(E \otimes L^{-1})$.

(b) Let $D \subset \mathbb{P}E$ be an irreducible curve, which is disjoint from $\Sigma_\infty$. Show that the class $[D]$ of $D$ in $H^2(\mathbb{P}E, \mathbb{Z})$ is $n(df + h)$, where $f$ is the class of the fiber, $h := c_1(\mathcal{O}_{\mathbb{P}E}(1))$, and $n := ([D], f)$. Conclude that the arithmetic genus of $D$ is $g(D) = d \left(\frac{n(n-1)}{2}\right) + n(g(C) - 1) + 1$.

Caution: In Proposition III.18 in Beauville’s text [B1] his $\mathcal{O}_S(1)$ is our $Q_{\mathbb{P}E}$.

2. Keep the notation of problem 1. Set $M := (p^*L) \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Following is an explicit construction of smooth curves in the linear system $|M^n|$, which are disjoint from $\Sigma_\infty$. Choose $b_i \in H^0(C, L^i)$, $0 \leq i \leq n$. Set $\beta := (b_0, b_1, \ldots, b_n)$ and $\alpha_i := p^*b_i$. Choose a section $\lambda_1$ of $H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(1))$, with divisor $\Sigma_\infty$ ($\lambda_1$ is unique, up to a scalar factor). If we identify $\mathcal{O}_{\mathbb{P}E}(1)$ with $\mathcal{O}_{\mathbb{P}E}(\Sigma_\infty)$, then $\lambda_1$ can be the section 1 of the latter. Choose a section $\lambda_0$ of $H^0(\mathbb{P}E, M)$, with divisor $\Sigma_0$. We get the section

$$\sigma_{\beta} := \sum_{i=0}^{n} \alpha_i \lambda_1^i \lambda_0^{n-i} \in H^0(\mathbb{P}E, M^n).$$

(1)

Denote by $\tilde{C}_\beta$ the divisor in $|M^n|$ corresponding to $\sigma_\beta$.

(a) Show that if $b_0 \neq 0$, then the intersection $\tilde{C}_\beta \cap \Sigma_\infty$ is empty.

(b) Show that if $b_0 \neq 0$, $b_i = 0$, for $1 \leq i \leq n-1$, and the divisor of $b_n$ in $|L^n|$ consists of $nd$ distinct points of $C$, then the curve $\tilde{C}_\beta$ is smooth and irreducible. Note: Points in a linear system, corresponding to smooth divisors, form a Zariski open subset (see Hartshorne’s Algebraic Geometry, Ch. I, section 5, Problem 5.15).

(c) Prove that $H^0(\mathbb{P}E, M^n)$ decomposes as the direct sum $\oplus_{i=0}^{n} \lambda_1^i \lambda_0^{n-i}p^*H^0(\mathbb{P}E, L^i)$. Conclude that every section of $H^0(\mathbb{P}E, M^n)$ is of the form given in Equation (1). Hint: It suffices to establish the direct sum decomposition

$$H^0(\mathbb{P}E, M^k) = \lambda_0H^0(\mathbb{P}E, M^{k-1}) \oplus \lambda_1^k p^*H^0(C, L^k),$$

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for all \( k \geq 1 \). Note first the isomorphism \( \sigma_0^*(M) \cong L \), and use it to construct the short exact sequence \( 0 \rightarrow M^{k-1} \xrightarrow{\lambda_0} M^k \xrightarrow{(\sigma_0)_*} (L^k) \rightarrow 0 \).

3. **Construction of projective spectral curves:** Keep the notation of problems 1 and 2. Let \( F \) be a locally free coherent sheaf of rank \( n \) over \( C \), \( \varphi : F \rightarrow F \otimes L \) a homomorphism of \( \mathcal{O}_C \)-modules, and \( p^*(\varphi) : p^*F \rightarrow p^*(F \otimes L) \) its pull-back to \( \mathbb{P}E \). Set

\[
\tilde{\varphi} := [p^*(\varphi) \otimes \lambda_1 - id_F \otimes \lambda_0] : p^*F \rightarrow (p^*F) \otimes M. \tag{2}
\]

Then the determinant \( \det(\tilde{\varphi}) \) is a section of \( M^n \). The divisor \( \tilde{C} \in |M^n| \) of \( \det(\tilde{\varphi}) \) is called the **spectral curve** of \( \varphi \).

(a) Show that the spectral curve \( \tilde{C} \) of \( \varphi \) is disjoint from \( \Sigma_{\infty} \).

(b) Set \( F := \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1} \). Let \( A \) be a matrix in \( V_{n,d} \) and \( \varphi : F \rightarrow F \otimes \mathcal{O}_{\mathbb{P}^1}(d) \) the associated homomorphism. Write \( char_A(x, \lambda) = \sum_{i=0}^n c_i(x) \lambda^{n-i} \). Set \( b_i := t_i^d c_i(t_i/t_0) \) and let \( b = (b_0, \ldots, b_n) \). Show that the spectral curve of \( \varphi \) is equal to the curve \( \tilde{C}_b \) constructed in \( \mathbb{P}_d := \mathbb{P}[\mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}] \) in problem 2.

(c) Let \( char : V_{n,d} \rightarrow |M^n| \) be the morphism sending a matrix \( A \) to its spectral curve (a divisor in the linear system on \( \mathbb{P}E \)). Show that the image of the morphism \( char \) contains the divisor of every curve considered in Question 2b.

(d) Prove Theorem 1.

4. Keep the notation above.

(a) Let \( \tilde{g} \) be the genus of the generic spectral curve in Theorem 1. Verify the equality

\[
\dim(V_{n,d}) = \tilde{g} + \dim |M^n| + \dim[PGL(n, \mathbb{C})].
\]

(b) The group \( GL(n, \mathbb{C}) \) acts on \( V_{n,d} \) by conjugation, and the action factors through \( PGL(n, \mathbb{C}) \). Show that the morphism \( char : V_{n,d} \rightarrow |M^n| \) is invariant under the \( PGL(n, \mathbb{C}) \)-action.

(c) Show that the co-kernel of the homomorphism \( \tilde{\varphi} \), given in Equation (2), is an \( \mathcal{O}_{\mathbb{P}E} \)-module, whose set-theoretic support is the spectral curve \( \tilde{C} \). The sheaf \( \tilde{F} := coker(\tilde{\varphi}) \otimes M^{-1} \) is a quotient of \( p^*F \). \( \tilde{F} \) is called the **eigen-line-bundle** of \( \varphi \). Prove the equality \( \chi(\tilde{F}) = \chi(F) \), where \( \chi \) is the sheaf cohomology bundle characteristic (on \( \mathbb{P}E \) and on \( C \)).

(d) Recall that \( p_*(p^*F) \cong F \otimes (p_*\mathcal{O}_C) \), by the projection formula. Let \( q : p^*F \rightarrow \tilde{F} \) be the quotient homomorphism. Prove that the composition

\[
\tilde{F} \xrightarrow{id \otimes 1} F \otimes p_*(\mathcal{O}_C) \cong p_*(p^*F) \xrightarrow{p_*(q)} p_*\tilde{F}.
\]

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\(^2\)If \( F = \bigoplus_{i=1}^n \mathcal{O}_C \) is the trivial vector bundle, then \( \tilde{\varphi} \) is an \( n \times n \) matrix, whose entries are sections of \( M \). The determinant \( \det(\tilde{\varphi}) \) is then the usual determinant, where we replace the product of \( n \) entries by their tensor product. For a general \( F \), the homomorphism \( \tilde{\varphi} \) induces a homomorphism

\[
\wedge^n \tilde{\varphi} : \wedge^n (p^*F) \rightarrow \wedge^n [(p^*F) \otimes M] \cong [\wedge^n (p^*F)] \otimes M^n.
\]

It corresponds to a section \( \det(\tilde{\varphi}) \) of \( M^n \), since \( \wedge^n (p^*F) \) is an invertible sheaf.
is an isomorphism. Hint: It suffices to prove injectivity, by part 4c. See Remark 2 for the meaning of this isomorphism.

**Remark 2** When \( \tilde{C} \) is smooth, the sheaf \( \tilde{F} \) is a locally free \( \mathcal{O}_{\tilde{C}} \)-module of rank 1, by part 4d. The isomorphism class of \( \tilde{F} \) determines the isomorphism class of the pair \((F, \varphi)\), and so the \( PGL(n, \mathbb{C}) \)-orbit of the matrix \( A \in V_{n,d} \), as follows. Let \( \mu : \tilde{F} \to \tilde{F} \otimes M \) be the homomorphism, given by tensoring with the section \( \lambda_0 \) of \( M \). The push-forward \( p_*(\mu) \) is equal\(^3\) to the homomorphism \( \varphi : F \to F \otimes L \), up to conjugation of \( \varphi \) by an automorphism of \( F \). Set \( \tilde{d} := \chi(\tilde{F}) + 1 - \tilde{g} \). The algebraic variety \( \text{Pic}^d(\tilde{C}) \), of degree \( \tilde{d} \) line-bundles on \( \tilde{C} \), is a \( \tilde{g} \)-dimensional smooth algebraic variety (Its dimension is equal to \( h^1(C, \mathcal{O}_C) \), by the discussion in Section I.10 of Beauville’s text on the exponential sequence \([B1]\)). Hence, the fiber \( \text{char}^{-1}(\tilde{C}) \) is an algebraic subset of \( V_{n,d} \) of dimension at most \( \tilde{g} + \dim PGL(n, \mathbb{C}) \). This must be exactly the dimension of the fiber, by part 4a. See \([BNR]\) for a detailed exposition.

5. Do problems 1, 2, 5, 6 in Chapter III page 37 of Beauville’s text \([B1]\).

**References**


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\(^3\)The above statement is due to the fact that a fiber of \( \tilde{F} \) over a point \( x \) of \( \tilde{C} \) is naturally identified with the \( x \)-eigen-line of the fiber \( \tilde{F}_{p(x)} \) of \( F \) over \( p(x) \), provided the eigenvalue \( x \) has multiplicity one (i.e., provided \( x \) is not a ramification point of \( \tilde{C} \to C \)). Furthermore, \( \mu \) acts on this fiber of \( \tilde{F} \) via tensorization with the corresponding eigenvalue \( x \in L_{p(x)} \). Finally, the fiber of \( p_*\tilde{F} \) over \( y \in C \) is naturally identified with the direct sum of the fibers of \( \tilde{F} \), over points in \( p^{-1}(y) \), provided \( y \) is not a branch points of \( \tilde{C} \to C \).