1. Let $S$ be a smooth connected complex projective surface, and $\epsilon : \hat{S} \to S$ the blow-up of a point $P \in S$. Let $D$ be a divisor in $\text{Div}(S)$. Show that the pull-back homomorphism $\epsilon^*: \text{Div}(S) \to \text{Div}(\hat{S})$ restricts to an isomorphism from $|D|$ onto $|\epsilon^*D|$. In particular, the linear system $\epsilon^*|D|$ is complete.

2. Let $C$ and $D$ be two linearly equivalent effective (and non-zero) divisors on a smooth connected complex projective surface $S$. Assume that $C \cap D$ consists of $n$ points $\{Q_i : 1 \leq i \leq n\}$, $n \geq 0$, at each of which the two curves intersect transversally. Let $P \subset |D|$, $i = 1, 2$. Let $f : S' \to S$ be the surface obtained by blowing-up each of the $n$ points of $C \cap D$. Denote by $E_i \subset S'$ the exceptional divisor over $Q_i$. Show that the fixed part of $f^*(P)$ is precisely $F := \sum_{i=1}^n E_i$ and that $P' := f^*(P) - F$ is base point free. Show that the associated morphism $\phi : S' \to (P')^*$ restricts to each of the exceptional divisors $E_i$ as an isomorphism from $E_i$ onto $(P')^*$.

3. (This is another take on Example II.14 (2) in Beauville’s text). Let $Q_1$, $Q_2$ be two distinct points in $\mathbb{P}^2$ and $C$ the line in $\mathbb{P}^2$ through $Q_1$ and $Q_2$. Let $P_i \subset |\mathcal{O}_{\mathbb{P}^2}(1)|$ be the pencil of lines through $Q_i$, $i = 1, 2$. Let $S$ be the blow-up of $\mathbb{P}^2$ at $Q_1$ and $Q_2$. Construct a morphism $\phi : S \to P_1^* \times P_2^*$. Show that $\phi$ is the blow-up of $P_1^* \times P_2^*$ at a point and that the strict transform of $C$ is the exceptional divisor of $\phi$.

4. Exercise II.20.1 page 23 in Beauville’s text. (Read Remark I.16 (i) page 8).

5. Let $C$ and $D$ be two distinct irreducible curves on a smooth connected complex projective surface $S$ and $x$ a point in $C \cap D$. Let $m_x(C)$ be the multiplicity of $C$ at $x$.

   (a) Let $\epsilon : \hat{S} \to S$ be the blow-up of $S$ at $x$, $E \subset \hat{S}$ the exceptional divisor, and $\hat{C}$ the strict transform of $C$. Show that $m_x(C) = \hat{C} \cdot E$.

   (b) Keep the notation of part 5a. Prove the equality

   $$m_x(C \cap D) = m_x(C)m_x(D) + \sum_{t \in (\hat{E} \cap \hat{D})} m_t(\hat{C} \cap \hat{D}) \quad (1)$$

   Conclude the inequality $m_x(C \cap D) \geq m_x(C)m_x(D)$. The latter is Axioms (5) of intersection multiplicities in Fulton’s Algebraic Curves, Chapter 3 section 3.

   (c) (Separating all points of intersection). Show that after a finite sequence of blow-ups, centered at points of $C \cap D$, and also at the points of intersections of the strict transforms of $C$ and $D$, we arrive at a surface $S'$, and a morphism $\pi : S' \to S$, satisfying: a) $\pi$ is an isomorphism over $S \setminus [C \cap D]$. b) The strict transforms $C'$ of $C$ and $D'$ of $D$ in $S'$ are disjoint. Hint: Note that Equation 1 implies the inequality $m_x(C \cap D) > \sum_{t \in (\hat{E} \cap \hat{D})} m_t(\hat{C} \cap \hat{D})$.

6. Exercise II.20.2 page 23 in Beauville’s text. This is a continuation of problem 5.