The field $k$ below is assumed algebraically closed.

1. Sketch the following curves in the affine plane $\mathbb{A}^2$: 
   
   $A := V(y - x^2), B := V(y^2 - x^3 + x), C := V(y^2 - x^3), D := V(y^2 - x^3 - x^2).$
   
   Let $P := (0, 0)$. Compute the six intersection numbers at $P$, of the six pairs of curves, and compare each to the product of the two multiplicities of the curves at $P$.

2. Prove the bilinearity property (6) of the local intersection multiplicity $I(P, C \cap D)$ for two affine plane curves (not necessarily irreducible, nor reduced, but rather subschemes of pure dimension 1). In other words, let $F, G_1, G_2 \in k[x, y]$ be polynomials of positive degree. Set $G := G_1G_2$. Assume that the algebraic subsets $V(F)$ and $V(G)$ do not have any common irreducible component. Prove the equality
   
   $$I(P, F \cap G) = I(P, F \cap G_1) + I(P, F \cap G_2).$$
   
   Hint: Let $\mathcal{O}_P$ be the local ring of $\mathbb{A}^2$ at $P$. Prove that the sequence
   
   $$0 \to \mathcal{O}_P/(F, G_2) \to \mathcal{O}_P/(F, G) \to \mathcal{O}_P/(F, G_1) \to 0$$
   
   is short exact, where $\psi(z + (F, G_2)) := G_1z + (F, G_1G_2)$.

   Remark: Observe that your argument goes through for pure one dimensional subschemes $C$ and $D$ over any smooth quasi-projective surface. (Beauville’s definition of the local intersection multiplicity considers only reduced curves, but in the question above we allow $G_1$ and $G_2$ to have common irreducible components, with arbitrary multiplicities, and $F$ may have irreducible components with arbitrary multiplicities).

3. Let $X$ be a smooth surface, $C$, $D$ curves on $X$, which do not have common irreducible components, and $P \in C$ a smooth point. Let $\mathcal{O}_{X,P}$ be the local ring of $X$ at $P$, $\mathcal{O}_{C,P}$ the local ring of $C$ at $P$, $f \in \mathcal{O}_{X,P}$ a local equation of $D$, and $\bar{f}$ its image in $\mathcal{O}_{C,P}$. Use the fact that localization is an exact functor to prove the following equality:
   
   $$m_P(C \cap D) = \text{ord}_p(\bar{f}).$$
   
   Conclude that when $C$ is smooth and irreducible, the restriction of the invertible sheaf $\mathcal{O}_X(D)$ to $C$ is isomorphic to $\mathcal{O}_C \left( \sum_{P \in C} m_P(C \cap D)P \right)$. (Compare, but do not use, Beauville, Lemma I.6).

4. (Hartshorne, Proposition 6.5. The proof is easy, so try to do it yourself and then check your answer). Let $X$ be a smooth quasi-projective variety, and $Z \subset X$ a closed algebraic subset. Set $U := X \setminus Z$. Prove the following statements.

   (a) The homomorphism $\rho : \text{Pic}(X) \to \text{Pic}(U)$, defined by
       
       $\sum n_i Y_i \mapsto \sum n_i(Y_i \cap U)$, is surjective. The divisor $Y_i \cap U$ above is the zero divisor of $U$, if the intersection is empty.
(b) If the codimension of $Z$ in $X$ is $\geq 2$, then $\rho$ is an isomorphism.

(c) If $Z$ is an irreducible subset of codimension 1, then the sequence

$$Z \to \text{Pic}(X) \xrightarrow{\rho} \text{Pic}(U) \to 0,$$

is exact, where the left homomorphism sends 1 to $Z$.

5. Let $X$ be a quasi-projective variety and $n \geq 1$. Show that $\text{Pic}(X \times \mathbb{P}^n)$ is isomorphic to $\text{Pic}(X) \times \mathbb{Z}$.

HINTS: i) Recall Hartshorne, Proposition II.6.6: Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. Then the homomorphism

$$\text{Pic}(X) \to \text{Pic}(X \times \mathbb{A}^1),$$

sending $\sum n_i Y_i$ to $\sum n_i (Y_i \times \mathbb{A}^1)$, is an isomorphism.

ii) Let $H \subset \mathbb{P}^n$ be a hyperplane, set $U := \mathbb{P}^n \setminus H$, and prove that the following sequence is short exact.

$$0 \to \mathbb{Z} \to \text{Pic}(X \times \mathbb{P}^n) \xrightarrow{\rho} \text{Pic}(X \times U) \to 0.$$

6. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Prove that $\text{Pic}(Q)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

HINT: Use question 5.

7. Let $C$ be a smooth projective curve of genus $g$. Set $X := C \times C$, and let $\Delta \subset X$ be the (reduced) diagonal curve $\{(P, P) : P \in C\}$. Show that the class of $\Delta$ in $\text{Pic}(X)$ belongs to the image of the homomorphism

$$\text{Pic}(C) \times \text{Pic}(C) \to \text{Pic}(X),$$

$$(D_1, D_2) \mapsto (D_1 \times C) + (C \times D_2),$$

if and only if $g = 0$.

HINT: Assume that $\Delta \sim (D_1 \times C) + (C \times D_2)$, $D_i \in \text{Pic}(C)$. Use Theorem I.4 in Beauville’s text to show that $\deg(D_i) = 1$. Then show that there exist two distinct points $P, Q \in C$, such that the divisors $P$ and $Q$ in $\text{Div}(C)$ are both linearly equivalent to $D_1$.

8. Let $C$ be a smooth projective curve of genus one. Fix a point $P_0 \in C$. Consider the map $a : C \to \text{Pic}(C)$, sending $P \in C$ to $P - P_0$. Let $\deg : \text{Pic}(C) \to \mathbb{Z}$ be the degree map, sending $\sum_{P \in C} n_P P$ to $\sum_{P \in C} n_P$. When the genus of $C$ is 1, we have shown in class, using the Riemann-Roch Theorem, that the image of $a$ is equal to the kernel of $\deg$. In particular, the choice of the point $P_0$ endows $C$ with a group structure. Recall also that the genus of a smooth curve of degree $d$ in $\mathbb{P}^2$ is $(d - 1)(d - 2)/2$.

Let $C$ be $V(zy^2 - x(x - z)(x - \lambda z)) \subset \mathbb{P}^2$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Set $P_0 := (0, 1, 0)$. Given a curve $D$ in $\mathbb{P}^2$, which does not contain $C$, denote by $C \cap D$ the divisor $\sum_{P \in C} m_P(C \cap D)P$ in $\text{Div}(C)$. 

2
(a) Set $H := V(z)$. Show that $m_{P_0}(H \cap C) = 3$. Conclude that the divisor class of $3P_0$ generates the image of the restriction homomorphism $\text{Pic}(\mathbb{P}^2) \to \text{Pic}(C)$.

(b) Let $P$, $Q$, and $R$ be points of $C$ (not necessarily distinct). Show that 
\[ a(P) + a(Q) + a(R) = 0 \text{ in } \text{Pic}(C), \]
if and only if there exists a line $L$ in $\mathbb{P}^2$, such that $(L \cap C) = P + Q + R$. The points $P$, $Q$, and $R$ are said in this case to be co-linear.

(c) Let $L_{P,Q}$ be the unique line in $\mathbb{P}^2$, such that $L_{P,Q} \cap C = P + Q + R$, for some point $R \in C$. (If $P = Q$, then $L_{P,Q}$ is the tangent line to $C$ at $P$). Show that $a(P) + a(Q) = a(S)$, if and only if $P_0$, $S$, and $R$ are co-linear, where $L_{P,Q} \cap C = P + Q + R$. Draw a picture.

(d) Give a geometric interpretation for the inversion $P \mapsto a^{-1}(-a(P))$. 