Homework 5

1. Read Section 2.3 in Huybrechts' book.

2. Show that $\mathcal{O}_{\mathbb{P}^n}(d)$ restricts to a projective line in $\mathbb{P}^n$ as a line bundle of degree d (Definition 2.3.29).

3. Do the following problems from Huybrechts, Section 2.3 page 89: 2.3.4, 2.3.5, 2.3.7, 2.3.8 (see hint below), 2.3.10 (see hint below).

4. Hint for 2.3.8: Let Y be a smooth hypersurface in a complex manifold X and p a point of Y. Use Lemma 2.3.22 to conclude that we have a short exact sequence of stalks at p
$$0 \to \mathcal{O}_X(-Y)_p \to \mathcal{O}_{X,p} \to \mathcal{O}_{Y,p} \to 0.$$
If, in addition, $\dim_{\mathbb{C}}(X) = 2$, and $f, g \in \mathcal{O}_{X,p}$ are germs of holomorphic functions, with g a local equation of Y, show that $\mathrm{ord}_p(f|_Y) = \dim_{\mathbb{C}}[\mathcal{O}_{X,p}/(f,g)]$, where (f,g) is the ideal genereted by f and g. Note that the right hand side of the latter equation is symmetric in f and g. If X is a compact complex surface and Y, Z two irreducible smooth divisors (complex curves) on X, conclude that $\deg(\mathcal{O}_X(Y)|_Z) = \deg(\mathcal{O}_X(Z)|_Y)$.

5. Hint for 2.3.10: Do the problem rigorously using algebraic methods and then give it a geometric description using Bezout's Theorem (Exercise 2.3.8) and intersections of lines through the point x with the degree 2 curve C. For the algebraic case where x does not belong to C show first that we may choose, without loss of generality, the point x to be any point not on C. Indeed, the canonical codomain of $\varphi_{\mathcal{O}_{\mathbb{P}^1}(2)}$ is $\mathbb{P}[H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*]$ and once we identify $\mathbb{P}^2$ with $\mathbb{P}[H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*] = \mathbb{P}[\mathrm{span}\{z_0^2, z_1z_2, z_2^2\}^*]$, the automorphism group $PGL[\mathrm{span}\{z_0, z_1\}^*] \cong PGL(2,\mathbb{C})$ of $\mathbb{P}^1$ gets identifies with the subgroup of the automorphism group $PGL[H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*]$ of $\mathbb{P}^2$ leaving the curve C invariant (the image in $PGL[H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*] \cong PGL(3)$ of the isometry group of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*$ preserving the quadratic polynomial Q defining C). This is a concrete version of the isomorphism $PGL(2,\mathbb{C}) \cong PSO(3,\mathbb{C})$. Now, $PSO(3,\mathbb{C})$ acts transitively on the open set $\mathbb{P}^2 \setminus V(Q)$.