Homework 4

A problem on Dolbeault cohomology. Let X be a connected compact complex manifold. Denote by $H^{p,q}_{Dol}(X)$ the Dolbeault cohomology of X (Def. 2.6.20 in Huybrechts' book, i.e., the quotient of the space $Z^{p,q}_{\bar{\partial}}(X)$ of $\bar{\partial}$-closed complex valued differential (p,q)-forms modulo the space $B^{p,q}_{\bar{\partial}}(X)$ of $\bar{\partial}$-exact (p,q)-forms). Denote by $H^{p,q}(X)$ the quotient of the space $Z^{p,q}_d(X)$ of d-closed differential (p,q)-forms modulo the space $B^{p,q}_d(X) := H^0(X, \mathcal{A}^{p,q}) \cap d[H^0(X, \mathcal{A}^{p+q-1})]$ of d-exact (p,q)-forms. When we will discuss Hodge theory (Ch. 3 in Huybrechts' book) we will get an isomorphism $H^{p,q}_{Dol}(X) \cong H^{p,q}(X)$, when X is Kähler. Here you will prove partial results that do not require Hodge theory, and use these to show that the Picard groups of compact complex tori are non-trivial.

1. Show that the inclusion $Z^{p,q}_d(X) \subset Z^{p,q}_{\bar{\partial}}(X)$ induces, for $p = 1$ and $q = 0$, a well defined injective homomorphism

$$f^{1,0} : H^{1,0}(X) \to H^{1,0}_{Dol}(X).$$

   Show that $f^{1,0}$ is an isomorphism if X is a compact torus.

2. Show that the inclusion $Z^{0,q}_d(X) \subset Z^{0,q}_{\bar{\partial}}(X)$ induces a well defined homomorphism

$$f^{0,q} : H^{0,q}(X) \to H^{0,q}_{Dol}(X). \tag{1}$$

3. Assume that $X := \mathbb{C}^n/\Lambda$ is an n-dimensional compact complex torus, $n > 0$, and let $\pi : \mathbb{C}^n \to X$ be the universal covering map.

   (a) Show that $H^{0,1}(X)$ is n-dimensional spanned by the descents $d\bar{z}_i$, $1 \leq i \leq n$, of the translation invariant $(0,1)$-forms on the universal cover.

   (b) Show that $f^{0,1} : H^{0,1}(X) \to H^{0,1}_{Dol}(X)$, given in (1), is injective. Hint: Show that if $\bar{\partial} g$ is d-closed for a function $g \in H^0(X, \mathcal{A}^0)$, then ∂g is a holomorphic $(1,0)$-form. Conclude, using part 3a, that $g \circ \pi$ is a polynomial of degree ≤ 1 in $\mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$, which is doubly periodic and hence constant.

   (c) Conclude that $\mathrm{Pic}(X)$ contains the quotient of an n-dimensional complex vector space by a free abelian group and is hence non-trivial.