Hint for problem 3.3.6 page 143 in Huybrechts' book

In Section 2.1 page 57 the space of n-dimensional complex tori $\mathbb{C}^n/\Lambda$ was described by varying the lattice $\Lambda \subset \mathbb{C}^n$. We showed that two complex tori $X_1 = \mathbb{C}^n/\Lambda_1$ and $X_2 = \mathbb{C}^n/\Lambda_2$ are isomorphic, if and only if there exists a linear automorphism of $\mathbb{C}^n$ mapping Λ_1 onto Λ_2. Thus, we can normalize the lattice Λ to be the span of the columns of an $n \times 2n$ matrix $\Pi := (Z, I)$, where I is the $n \times n$ identity matrix and Z is an $n \times n$ matrix with $Im(Z)$ invertible. Denote by $\mathcal{M}_n$ the space of $n \times n$ complex matrices Z with invertible imaginary part. Denote by Λ_Z the lattice spanned over $\mathbb{Z}$ by the columns of (Z, I). Write a $2n \times 2n$ invertible matrix G with integers entries in the form $G = \begin{pmatrix} G_{1,1} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{pmatrix}$ with $n \times n$ blocks $G_{i,j}$. It is easy to show that the isomorphism class of an n-dimensional compact complex torus $X = \mathbb{C}^n/\Lambda_Z$ is represented by the $GL(2n, \mathbb{Z})$-orbit

$$\{(ZG_{1,2} + G_{2,2})^{-1}(ZG_{1,1} + G_{2,1}) \ : \ G \in GL(2n, \mathbb{Z})\}$$

in $\mathcal{M}_n$.

An element ω in $H^2(X, \mathbb{Z})$ corresponds to an alternating $\mathbb{Z}$-valued bilinear form on the lattice Λ. Denote by $\beta := \{\lambda_1, \dots, \lambda_{2n}\}$ the basis of Λ consisting of columns of $\Pi := (Z, I)$. Then ω is given by an anti-symmetric $2n \times 2n$ matrix with integer entries of the form $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$, with $A = -A^t$ and $C = -C^t$.

1. Show that an element $\omega \in H^2(X, \mathbb{Z})$ belongs to $H^{1,1}(X, \mathbb{Z})$, if and only if

$$A - BZ + Z^t B^t + (Z^t)CZ = 0. \tag{1}$$

Hint: Recall that ω is of type $(1, 1)$, if and only if

$$\omega(iv_1, iv_2) = \omega(v_1, v_2), \tag{2}$$

for all $v_1, v_2 \in \mathbb{C}^n$. Let $J : \mathbb{C}^n \to \mathbb{C}^n$ be the complex structure (multiplication by i). Let R be the $2n \times 2n$ matrix of J in the basis β of $\mathbb{C}^n$ as a vector space over $\mathbb{R}$. Then (2) is equivalent to

$$R^t \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} R = \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}. \tag{3}$$

The coordinate linear transformation $[\ ]_\beta : \mathbb{C}^n \to \mathbb{R}^{2n}$ satisfies $[i\lambda_j]_\beta = R_j$, where R_j is the j-th column of R. Now left multiplication $\Pi : \mathbb{R}^{2n} \to \mathbb{C}^n$ is the inverse of $[\ ]_\beta$. Hence, $i\lambda_j = \Pi R_j$ and so $i\Pi = \Pi R$. Conclude that

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} = \begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix} R. \tag{4}$$

Now show that (1) is equivalent to the conjunction of (3) and (4). Note that $J^{-1} = -J$ and that $\begin{pmatrix} \Pi \\ \bar{\Pi} \end{pmatrix}$ is invertible.

2. When $n = 2$ and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & d \\ e & f \end{pmatrix}$, and $C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, $a, b, c, d, e, f \in \mathbb{Z}$, conclude that ω belongs to the Neron-Severi group of X, if and only if the period matrix (Z, I) satisfies the quadratic equation

$$a + ez_{1,1} - bz_{1,2} + fz_{2,1} - dz_{2,2} + c\det(Z) = 0.$$

3. Assume $n = 2$. Show that if $X = \mathbb{C}^n/\Lambda_Z$ and the Neron Severi group $NS(X)$ is non-trivial, then Z belongs to a countable union of co-dimensional 1 complex hypersurfaces in $\mathcal{M}_n$. Conclude that for a general point Z of $\mathcal{M}_n$, the complex torus $X = \mathbb{C}^n/\Lambda_Z$ has a trivial Neron-Severi group.