Review:

Def: Let X, Y be complex manifolds.
A map $f : X \to Y$ is holomorphic (a morphism) if for any chart (U, φ) of X and any chart (U', φ') of Y

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & Y \\
 & \cup\ \text{open} & & \cup\ \text{open} \\
U \cap f^{-1}(U') \subset & U \arrow[d, "\varphi"', "\cong"] & & U' \arrow[d, "\varphi'", "\cong"'] \\
\varphi(U \cap f^{-1}(U')) \subset \arrow[rrru, bend right] & \varphi(U) & & \varphi'(U') \\
 & \cap\ \text{open} & & \cap\ \text{open} \\
 & \mathbb{C}^n & & \mathbb{C}^m
\end{tikzcd}$$

$\varphi' \circ f \circ \varphi^{-1}$ is holomorphic.

Def: Let X be a complex manifold. We have the sheaf $\mathcal{O}_X$ of holomorphic functions. To every open subset $U \subset X$ we get the ring $\mathcal{O}_X(U)$ of holo functions on U.

Given a point $p \in X$, we get the stalk $\mathcal{O}_{X,p}$ of $\mathcal{O}_X$ at p

$\varinjlim_{\{U : p \in U\}} \mathcal{O}_X(U)$. In terms of charts

$$p \in \underset{\substack{\cap \text{ open} \\ X}}{U} \xrightarrow{\varphi} \underset{\substack{\Psi \\ \varphi(p)}}{\varphi(U)} \underset{\text{open}}{\subset} \mathbb{C}^n$$

$\mathcal{O}_{X,p}$ "is" just $\mathcal{O}_{\mathbb{C}^n, \varphi(p)}$ = the ring of convergent power series centered at $\varphi(p)$.

The stalk $\mathcal{O}_{X,p}$ is a local integral domain, which is a noetherian UFD. We get its quotient field $Q(\mathcal{O}_{X,p})$.

Def: Let X be a complex manifold. A meromorphic function is a function

$$f : X \dashrightarrow \bigcup_{p \in X} Q(\mathcal{O}_{X,p})$$
$$p \longmapsto f_p$$

satisfying for every $p \in X$ there is an open neighborhood U of p and holo functions g, h on U, with h non-zero on every connected component of U, such that

$$f_{\tilde{p}} = \frac{g_{\tilde{p}}}{h_{\tilde{p}}} \quad \text{for all } \tilde{p} \in U.$$

Observe: If X is a connected cpx manifold, then the ring $K(X)$ of meromorphic function on X is a field.

Thm: 2.1.9: Let X be a cpt, connected cpx manifold of dim n. Then the following hold.

1) (Seigel's Thm) $\operatorname{tr\,deg}_{\mathbb{C}}(K(X)) \leq n$.

I.e., for every $n+1$ meromorphic functions

$b_1, \dots, b_{n+1} \in K(X)$, there exists a polynomial $F(x_1, \dots, x_{n+1}) \in \mathbb{C}[x_1, \dots, x_{n+1}]$ such that $\underbrace{F(b_1, \dots, b_{n+1})}_{\equiv 0} \in K(X)$ is identically zero.

2) (Remmert) Let $a(X) \underset{\text{def}}{:=} \operatorname{trdeg}_{\mathbb{C}}(K(X))$.

Choose $b_1, \dots, b_{a(X)} \in K(X)$ which are algebraically independent (so $\mathbb{C}(b_1, \dots, b_{a(X)}) \simeq \mathbb{C}(x_1, \dots, x_{a(X)})$)

Then $K(X)$ is a finite dimensional $\mathbb{C}(b_1, \dots, b_{a(X)})$ vector-space.

$[K(X) : \mathbb{C}(b_1, \dots, b_{a(X)})]$ is a finite field extension.

<u>Remark:</u> $K(\underset{\mathbb{P}^n}{\mathbb{C}\mathbb{P}^n}) = \mathbb{C}(z_1, \dots, z_n)$

$K(\mathbb{C}^n) \not\supseteq \mathbb{C}(z_1, \dots, z_n)$

<u>Ex:</u> $(n=1)$ $K(\mathbb{C}\mathbb{P}^1) = \mathbb{C}(z)$

$e^z \in K(\mathbb{C}^1)$, $e^z \notin \mathbb{C}(z)$.

Examples of cpx manifolds:

As a set $\mathbb{P}^n$ = lines in $\mathbb{C}^{n+1}$ through 0 $= \mathbb{C}^{n+1}\setminus\{0\} \big/ \mathbb{C}^\times$

at least one $z_i \neq 0$

$$q : \mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{P}^n$$

$$(z_0, \dots, z_n) \longmapsto (z_0 : z_1 : \cdots : z_n)$$

homogeneous coordinates

If $\lambda \in \mathbb{C}^\times$, then

$$(\lambda z_0 : \lambda z_1 : \dots : \lambda z_n) = (z_0 : \quad : z_n)$$

The topology of $\mathbb{P}^n$ is the quotient topology, i.e. a subset $U \subset \mathbb{P}^n$ is open, if and only if $q^{-1}(U)$ is open in $\mathbb{C}^{n+1}\setminus\{0\}$.

<u>Note</u>: $\mathbb{P}^n$ is compact, because q maps the sphere $\{z : \sum_{i=0}^n |z_i|^2 = 1\}$ onto $\mathbb{P}^n$.

An holo atlas for $\mathbb{P}^n$:

$$U_i = \{(z_0 : \dots : z_n) : z_i \neq 0\}$$

$$\varphi_i : U_i \xrightarrow{\sim} \mathbb{C}^n$$

$$[z_0 : \cdots : z_n] \longmapsto \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right)$$

$$[z_0 : \cdots : z_n] \overset{\tilde{\varphi}_i}{\longmapsto} \left(\frac{z_0}{z_i} : \cdots : \frac{z_i}{z_i} : \cdots : \frac{z_n}{z_i}\right) \in \mathbb{C}^{n+1}$$

$$\begin{array}{ccc}
U_i & \xrightarrow{\varphi_i} & \mathbb{C}^n \subset \mathbb{C}^{n+1} \\
\cup & & \cup \\
U_i \cap U_j & & \varphi_i(U_i \cap U_j) \\
 & & \downarrow \varphi_j \circ \varphi_i^{-1} \\
U_i \cap U_j & & \varphi_j(U_i \cap U_j) \\
\cap & & \cap \\
U_j & \xrightarrow{\varphi_j} & \mathbb{C}^n \subset \mathbb{C}^{n+1}
\end{array}$$

$U_i \xrightarrow{\tilde{\varphi}_i} \mathbb{C}^{n+1}$, $\quad U_j \xrightarrow{\tilde{\varphi}_j} \mathbb{C}^{n+1}$

$$[z_0 : \cdots : z_n] \overset{\tilde{\varphi}_j}{\longmapsto} \left(\frac{z_0}{z_j} : \cdots : \frac{z_j}{z_j} : \cdots : \frac{z_n}{z_j}\right), \quad \frac{z_j}{z_j} = 1$$

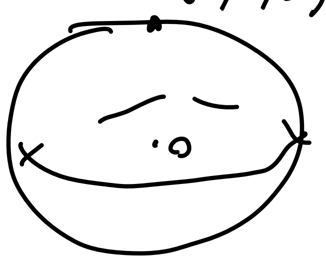
$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ is multip by (z_i/z_j)

Ex: Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$f(x,y,z) = 1 = f(z_0)$$

$z_0 = (0,0,1)$



$= X$

At z_0 $\quad \left.\frac{\partial f}{\partial z}\right|_{z_0} = 2z\big|_{z_0} = 2 \neq 0.$

Note that the upper hemisphere (an open neighborhood of z_0 in X)

is the graph of $g(x,y) = \sqrt{1-x^2-y^2}$

$$\{(x,y,g(x,y)) ; 0 \le x^2+y^2 < 1\}$$

<u>Theorem:</u> (Implicit function Thm)

Let $U \subset \mathbb{C}^m$ be an open subset.

Let $f : U \to \mathbb{C}^n$ be a holomorphic function, $= (f_1, \dots, f_m)$ where $m \geq n$.

Suppose that $\det\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{1 \le i,j \le n} \ne 0 \quad \forall p \in U$.

$n \times n$

Then locally around a point $z_0 \in U$, $z_0 = (a_1, \dots, a_m)$

the subvariety $\{z : f(z) = f(z_0)\}$ $X = Z \cap U$

is the graph of a function

$(z_1, \dots, z_n) = g(z_{n+1}, \dots, z_m)$.

$= (g_1(z_{n+1}, \dots, z_m), g(\ ), \dots, g_n(\ ))$

In other words, there is an open neigh $U_1 \subset \mathbb{C}^{m-n}$ and an open set

$\mathcal{U}_2 \subset \mathbb{C}^n$, such that $z_0 \in \mathcal{U}_1 \times \mathcal{U}_2 \subset \mathcal{U}$ and a holo map $g : \mathcal{U}_1 \to \mathcal{U}_2$, such that $X \cap (\mathcal{U}_1 \times \mathcal{U}_2) = \{ z \in \mathcal{U}_1 \times \mathcal{U}_2 ; f(z) = f(z_0) \}$

$z \in$ if and only if $(z_1, \dots, z_n) = g(z_{n+1}, \dots, z_m)$.

$z = (z_1, \dots, z_m)$

Proof: The IFT from real analysis yield the existence of $\mathcal{U}_1, \mathcal{U}_2$ and g as above, with g only C^∞-function. We need to prove that g is holomorphic.

We have

$$f\big(\underbrace{g_1(z_{n+1}, \dots, z_m), g_2(\,), \dots, g_n(\,), z_{n+2}, \dots, z_m}_{z_1}\big) \equiv f(z_0)$$

for all $(z_{n+2}, \dots, z_m) \in \mathcal{U}_1$

So

$$0 = \frac{\partial f_i(g_1(\,), \dots, g_n(\,), z_{n+2}, \dots, z_m)}{\partial \bar{z}_j} \quad \text{for } n+1 \le j \le m$$

$\sim \bar{w}_{j-n}$

Let $w_1 = z_{n+2}, \dots \to w_{m-n} = z_m$

Set $g_k := \begin{cases} \text{the old } g_k, & 1 \le k \le n \\ z_k, & n+1 \le k \le m \end{cases}$

$G = (g_1, \dots, g_m) : \mathbb{C}^{m-n} \longrightarrow \mathbb{C}^m$

$$df \circ dG = 2n\left\{ \underbrace{\begin{pmatrix} \left(\frac{\partial f_i}{\partial z_j}\right) & 0 \\ 0 & \frac{\partial \bar f_i}{\partial \bar z_j}\end{pmatrix}}_{2n} \right. \circ \underbrace{\begin{pmatrix} \frac{\partial g_i}{\partial w_j} & \frac{\partial g_i}{\partial \bar w_j} \\ \frac{\partial \bar g_i}{\partial w_j} & \frac{\partial \bar g_i}{\partial \bar w_j}\end{pmatrix}}_{2(m-n)} \left.\vphantom{\begin{pmatrix} \frac{\partial g_i}{\partial w_j} \\ \frac{\partial \bar g_i}{\partial w_j}\end{pmatrix}}\right\} 2n$$

$\frac{\partial f_i}{\partial \bar w_{j-n}}$ = the $(i, m-n+(j-n))$ entry

$$\underbrace{\frac{\partial f_i}{\partial \bar z_j}}_{\bar w_{j-n}} = \sum_{k=1}^{m} \frac{\partial f_i}{\partial z_k} \cdot \frac{\partial g_k}{\partial \bar w_{j-n}} + \sum_{k=1}^{m} \underbrace{\frac{\partial f_i}{\partial \bar z_k}}_{0} \circ \frac{\partial \bar g_k}{\partial \bar w_{j-n}}$$

$$= \underline{\sum_{k=1}^{n} \frac{\partial f_i}{\partial z_k} \frac{\partial g_k}{\partial \bar w_{j-n}}} + \sum_{k=n+1}^{m} \underbrace{\frac{\partial \overset{g_k}{\overset{\shortparallel}{z_k}}}{\partial \underset{\bar w_{j-n}}{\underset{\shortparallel}{\bar z_j}}}}_{=0}$$

since z_k is holo

= the i-th entry in

$$\underbrace{\left(\frac{\partial f_i}{\partial z_k}\right)_{1 \le i, k \le n}}_{\text{invertible}} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial \bar z_j} \\ \vdots \\ \frac{\partial g_n}{\partial \bar z_j} \end{pmatrix}$$

So the i-th entry of the above product is 0 for all i and all $n+1 \leq j \leq m$. So g_k is holo for $1 \leq k \leq n$. $\square$

<u>Ex:</u> Affine hypersurfaces, which are <u>cpx manifolds:</u>

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function, such that $Jf = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$ does not vanish on $\underbrace{Z(f)}_{X}$. Then $X = Z(f)$ is a complex manifold.

<u>Ex:</u> $n = 2$, $f(z_1, z_2) = z_1^2 + z_2^2 - 1$

Then $Jf = (2z_1, 2z_2)$

Proof that X is a cpx manifold of dimension $n-1$. We need to prove a holomorphic atlas.

<u>Charts:</u> If $p \in Z(f)$ and $\frac{\partial f}{\partial z_j}(p) \neq 0$

Then there exist an open neigh U of p in $\mathbb{C}^n$ and open sets $U_1 \subset \mathbb{C}^{n-1}$ and $U_2 \subset \mathbb{C}$

$z_1, \dots, \hat{z}_j, \dots z_n$ $\quad$ z_j

such that $p \in U_1 \times U_2 \subset U$ and a holo function $g : U_1 \to U_2$

such that $f(z_1, \dots z_{j-1}, g(z_1, \hat{z}_j, z_n), z_{j+1}, \dots, z_n) = f(p)$

$U_1 \times U_2$ is an open set in the chart

and $\varphi : U_1 \times U_2 \longrightarrow U_1 \subset \mathbb{C}^{n-1}$

is just the projection.

Then φ is holo.

The gluing transformation will be biholomorphic, since g above is holomorphic.

Projective hypersurfaces: (in $\mathbb{P}^n$)

Let f be a homogeneous poly of degree $d > 0$ in $z_0, \dots, z_n$

Set $X = V(f) = \{(z_0 : \dots : z_n) \in \mathbb{P}^n : f(z_0, \dots, z_n) = 0\}$

Well defined

$\lambda \in \mathbb{C}^\times$

$$f(\lambda z_0 : \dots : \lambda z_n) = \underset{\underset{0}{\#}}{\lambda^d} f(z_0 : \dots : z_n).$$

Assume that $V\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right) = \emptyset$

Then X is a compact complex manifold.

Note:

$$\boxed{\sum_{k=0}^{n} z_k \frac{\partial f}{\partial z_k} = d \cdot f} \quad (*)$$

Reason: f is homog, so sum of monomials of the form $\underbrace{z_0^{e_0} \cdots z_n^{e_n}}_{M(z)}$ where $\sum_{k=0}^{n} e_k = d$.

$$\sum_{k=0}^{n} z_k \frac{\partial (z_0^{e_0} \cdots z_n^{e_n})}{\partial z_k} = \sum_{k=0}^{n} e_k z_k \underbrace{(z_0^{e_0} \cdots z_{k-1}^{e_{k-1}} z_k^{e_k - 1} \cdots z_n^{e_n})}_{M(z)}$$

$= \underbrace{\left(\sum_{k=0}^{n} e_k \right)}_{d} \mu(z).$

<u>Note</u>: It suffices to show that $X \cap (z_j \neq 0)$ is an affine cpx manifold.

<u>Take $j = 0$. (W.L.O.G)</u>

<u>Claim</u>: In the chart $(z_0 \neq 0)$ of $\mathbb{P}^n$ we have that $X \cap (z_0 \neq 0) =$
$f(1, z_1, \ldots, z_n) = 0$, where now $z_1, \ldots, z_n$ are affine coor of $\mathbb{C}^n$.

It suffices to show that If $p \in X \cap (z_0 \neq 0)$ then $\frac{\partial f}{\partial z_j}(p) \neq 0$ for some $\underline{\underline{1}} \leq j \leq n$.

<u>Pf</u>: Assume otherwise that $\frac{\partial f}{\partial z_j}(p) = 0$, for $1 \leq j \leq n$, Then $\frac{\partial f}{\partial z_0}(p) \neq 0$, by our assumption on f.

$P = (\underbrace{a_0}_{\neq 0} : \cdots : a_n)$

$$\underbrace{a_0}_{\neq 0} \underbrace{\frac{\partial f}{\partial z_0}(p)}_{\neq 0} + \underbrace{a_1 \frac{\partial f}{\partial z_1}(p)}_{0} + \cdots + a_n \underbrace{\frac{\partial f}{\partial z_n}(p)}_{0} = d \underbrace{f(p)}_{0}$$

The LHS $\neq 0$, the RHS $= 0$. A contradiction. $\square$

Ex: We can compactify the affine conic $z_1^2 + z_2^2 - 1 = 0$ to a smooth 1-dim'l cpx submanifold of $\mathbb{P}^2$ by

$$V(z_1^2 + z_2^2 - z_0^2).$$