Let $U \underset{\text{open}}{\subset} \mathbb{C}^n$, $I: TU \to TU$ the complex structure, and let g be a Riemannian metric on U compatible with I. The FUNDAMENTAL FORM associated to (I, g) is

$$\boxed{\omega := g(I(\cdot), \cdot)}$$

It is a real $(1,1)$-form.

Remark: Any two of I, g, ω determine the third.

Def: If ω is closed, $d\omega = 0$, then g is called a Kähler metric and ω is a Kähler form.

Ex: For the standard metric we get

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j.$$

Def: Let W be a complex vector space. A pairing $H: W \times W \longrightarrow \mathbb{C}$ is a Hermitian form, if

1) $H(w, w) > 0$, $\forall w \in W$ such that $w \neq 0$

2) $H(cw, v) = c H(w, v)$, $\forall w, v \in W$, $c \in \mathbb{C}$

3) $H(v, w) = \overline{H(w, v)}$.

Ex: $W = \mathbb{C}^n$, $H(v, w) = (v^t) M \bar{w}$, where M is an $n \times n$ matrix with $\mathrm{Re}(M)$ symmetric and positive definite and $\mathrm{Im}(M)$ anti-symmetric and non-degenerate.

Lemma: Let $(V, (,), I)$ be a real vector space with compatible $(,)$ and I and fundamental form $\omega = (I(\cdot), \cdot)$. Then $H: V \times V \to \mathbb{C}$, given by $H(v, u) = (v, u) - i \underbrace{\omega(v, u)}_{(I(v), u)}$ is a Hermitian form.

Cor: Given $U \overset{\text{open}}{\subset} \mathbb{C}^n$ and a metric g on U, compatible with I, with fundamental form ω we get a Hermitian form

$$\boxed{h := g - i\omega}$$

In local coordinates $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$ is a basis for $(T_z U, I)$. Set

$$h_{ij}(z) := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) - i\,\underbrace{\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)}_{g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_j}\right)}$$

A long calculation yields:

$$\boxed{\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij}\, dz_i \wedge d\bar{z}_j}$$

Def: We say that the metric g osculates at the origin $0 \in U \overset{\text{open}}{\subset} \mathbb{C}^n$ to the standard metric to order 2 if $(h_{ij}) = id + \underbrace{O(|z|^2)}$

means that the 1st order partials vanish at 0.

$\frac{\partial h_{ij}}{\partial z_k}$ and $\frac{\partial h_{ij}}{\partial \bar{z}_n}$

Prop: Let $U \overset{open}{\subset} \mathbb{C}^n$, $I : TU \to TU$ the cpx str, g a compatible metric. Then the fundamental form ω is closed $d\omega = 0$, if and only if at every point $a \in U$, there are local coor

$$\begin{array}{ccc} 0 & \longmapsto & a \\ \cap & & \cap \\ U' & \xrightarrow{f} & U \\ \cap^{open} & & \cap \\ \mathbb{C}^n & & \mathbb{C}^n \end{array}$$

where f is biholomorphic, and $f^*(\omega)$ osculates to order 2 at $0 \in U'$ the standard metric.

Pf: ($\Leftarrow$) Given such biholo f,

$$(d\omega)(a) = 0 \iff d(f^*\omega)(0) = 0.$$

But if $f^*\omega = \frac{i}{2} \sum_{i,j=1}^n h_{ij}\, dz_i \wedge d\bar{z}_j$

and $\frac{\partial h_{ij}}{\partial z_k} = 0 \ \forall k$ and $\frac{\partial h_{ij}}{\partial \bar{z}_k} = 0 \ \forall k$

then $d(f^*\omega) = \frac{i}{2}\Big[\sum_{i,j=1}^n \sum_{k=1}^n \frac{\partial h_{ij}}{\partial z_k}\Big|_0 dz_k \wedge dz_i \wedge d\bar{z}_j$

$+ \sum_{i,j=1}^n \sum_{k=1}^n \frac{\partial h_{ij}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j \Big]$

so $d(f^*\omega)(a) = 0$,

So $(d\omega)(a) = 0 \quad \forall a \in U$.

($\Leftarrow$) Assume that $d\omega = 0$.

The new coor (components of f^{-1}) $\xi_1, \dots, \xi_n$ are expressed as functions of $z_1, \dots, z_n$.

We need $(f^{-1})^*\left(\frac{i}{2}\sum_{j=1}^{n} d\xi_j \wedge d\bar{\xi}_j\right) = \omega$

at a

to order 2

W.M.A that $a = 0$ by translation.

W.M.A that $h_{ij} = \delta_{ij} + \sum_{k=1}^{n} a_{ijk} z_k + \sum_{k=1}^{n} a'_{ijk} \bar{z}_k +$ terms of higher order.

$\boxed{\omega = \frac{i}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j}$

after a linear change of coordinates,

so $a_{ijk} = \dfrac{\partial h_{ij}}{\partial z_k}$ and $a'_{ijk} = \dfrac{\partial}{\partial \bar{z}_k} h_{ij}$

a_{ijk}

''

$$0 = d\omega = \frac{i}{2} \sum_{i,j} \left[\sum_{k=1}^{n} \overbrace{\frac{\partial h_{ij}}{\partial z_k}} \, dz_k \wedge dz_i \wedge d\bar{z}_j + \sum_{k=1}^{n} \underbrace{\frac{\partial h_{ij}}{\partial \bar{z}_k}}_{a'_{ijk}} \, d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j \right] + \cdots$$

So $0 = a_{ijk} \, dz_k \wedge dz_i \wedge d\bar{z}_j + a_{kji} \, dz_i \wedge dz_k \wedge d\bar{z}_j$

So $\boxed{a_{ijk} = a_{kji}}$ (*)

Similarly

$$\boxed{a'_{ijk} = a'_{ikj}} \quad (**)$$

$$\omega = \bar{\omega} \implies \overline{h_{ij}} = h_{ji} \implies$$

$$\frac{\partial h_{ij}}{\partial \bar{z}_k} = \frac{\partial \overline{h_{ji}}}{\partial \bar{z}_k} = \overline{\left(\frac{\partial h_{ji}}{\partial z_k} \right)} = \bar{a}_{jik}$$

$\overset{\text{def}}{=}$ a'_{ijk}

$$\boxed{a'_{ijk} = \bar{a}_{jik}} \quad (***)$$

Define the new coordinates by

$$\xi_j = z_j + \frac{1}{2}\sum_{i,k=1}^{n} a_{ijk} z_i z_k$$

$$d\xi_j = dz_j + \frac{1}{2}\sum_{i,k=1}^{n} a_{ijk} z_k dz_i + \overset{\cancel{a_{ij}}\ a_{kji}}{z_i\, dz_{\cancel{k}}}$$

$$= dz_j + \sum_{i,k=1}^{n} a_{ijk} z_k dz_i$$

$$d\overline{\xi_j} = d\overline{z_j} + \sum_{i,k=1}^{n} a'_{jik}\overline{z_k}\, d\overline{z_i}$$

Now compute:

$$\frac{i}{2}\sum_{j=1}^{n} d\xi_j \wedge d\overline{\xi_j} = \cdots = \omega$$

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij}\, dz_i \wedge d\bar z_j$$

to first order.

$\square$

Complex Manifolds:

Def: 1) An n-dim'l complex manifold X is a Hausdorff topological space admitting a countable basis (of open sets) which is endowed with an equivalence class of holomorphic atlases

2) A holomorphic atlas is an open covering $X = \bigcup_{i \in I} U_i$

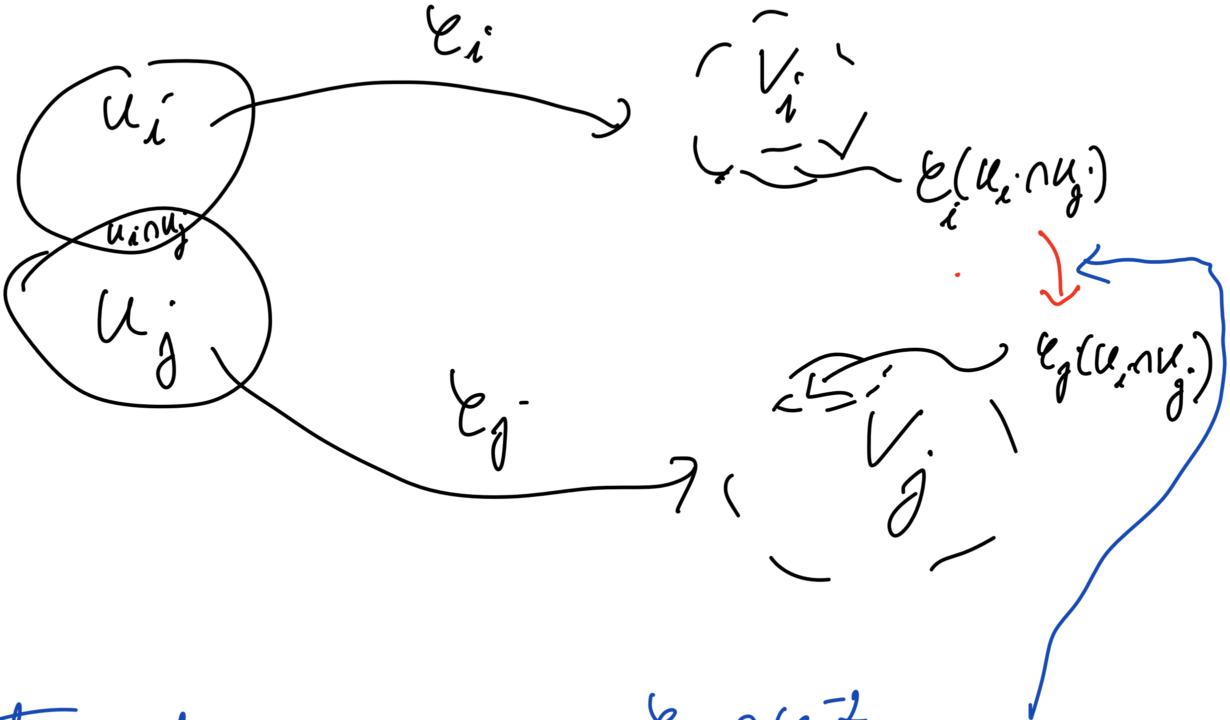
and homeomorphismes

$$\varphi_i : U_i \xrightarrow{\simeq} V_i \overset{\text{open}}{\subset} \mathbb{C}^n$$

such that the transition functions are biholomorphic

$f|_{U_i}$ $\quad$ $f \circ \varphi_i^{-1} \,|\, \varphi_i(U_i)$

$\mathbb{C}$



Transition func $= \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)}$

3) Two holomorphic atlases are equivalent if their union is a holomorphic atlas.

Def: Let X be a cpx manifold, $\{(U_i, \varphi_i)\}_{i \in I}$ a holo atlas.

A function $f: X \to \mathbb{C}$ is holo, if $f|_{U} \circ \varphi_i^{-1}|_{\varphi_i(U_i)} : \varphi_i(U_i) \to \mathbb{C}$ is holo for all $i \in I$.

$\varphi_i(U_i) \underset{\text{open}}{\subset} \mathbb{C}^n$

Identity Thm: Let X be a connected cpx manifold and $f, g : X \to \mathbb{C}$ two holo functions. If $f(z) = g(z)$, for all z in some non-empty open subset of X, then $f = g$ globally on X.

Pf: The condition that the coeff of the Taylor series of f and g are all equal is an open condition, by Taylor's Thm and closed. So if this locus is non-empty (as assumed), then it is the whole X, since X is connected. $\square$

Prop: Let X be a compact and connected complex manifold. Every holomorphic function $f : X \to \mathbb{C}$ is constant.

Pf: Since X is compact, $|f|$ has a maximum in X, hence a local maximum at some $a \in X$.

So f is constant in some open neigh of a chart U_i containing a, by the Maximum principle for subsets $\varphi_i(U_i) \subset \mathbb{C}^n$ open, so there exists a non-empty subset $W \subset X$ such that $f \equiv c$ on W, so $f \equiv c$ on X, by the identity theorem.

$\square$

Def: Let X, Y be two complex manifolds. A map $f: X \to Y$ is holomorphic (a morphism) if for any chart (U, φ) of X and (U', φ') of Y

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\text{open } \cup & & \cup \\
U \supset f^{-1}(U')\cap U & & U' \\
\varphi \downarrow & & \downarrow \varphi' \\
\varphi(U) \supset \varphi(f^{-1}(U')\cap U) & \longrightarrow & \varphi'(U') \\
\cap \text{ open} & & \cap \text{ open} \\
\mathbb{C}^n & & \mathbb{C}^k
\end{array}
$$

$\varphi' \circ f \circ \varphi^{-1}\big|_{\varphi(f^{-1}(U')\cap U)}$

$\varphi' \circ f \circ \varphi^{-1}$ is holomorphic.