Ex: Let $\pi : X \to Y$ be a submersive holomorphic map between two cpx manifolds $f : Y \to Y$, $\tilde{f} : X \to X$ automorphisms. We get

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

is commutative.

$$p = f \circ \pi \circ (\tilde{f})^{-1}$$

We get the commutative

$$\begin{tikzcd} 0 \arrow[r] & \ker(d\pi) \arrow[r] \arrow[d, dashed] & TX \arrow[r, "d\pi"] \arrow[d, "d\tilde{f}", "\simeq"'] \arrow[dr, bend left, "d(f\circ\pi)"] & \pi^*TY \arrow[r] \arrow[d, "\pi^*(df)", "\simeq"'] & 0 \\ 0 \arrow[r] & \tilde{f}^*(\ker(dp)) \arrow[r] & \tilde{f}^*TX \arrow[r] \arrow[dr, "\tilde{f}^*(dp)"'] & \pi^*(f^*TY) \arrow[r] \arrow[d, equal] & 0 \\ & & & \tilde{f}^* p^* TY & \end{tikzcd}$$

In particular, $d\tilde{f}$ restricts to an isomorphism of v.b.
$d\tilde{f}| : \ker(d\pi) \to \tilde{f}^*(\ker(dp))$.

Ex: Let $\ell \in \mathbb{P}(V)$, $a_\ell : GL(V) \to \mathbb{P}(V)$, $g \mapsto g(\ell)$. We get for $f \in GL(V)$,

$$\begin{tikzcd} GL(V) \arrow[r, "a_\ell"] \arrow[d, "Ad_f"'] & \mathbb{P}(V) \arrow[d, "f"] \\ GL(V) \arrow[r, "a_{f(\ell)}"'] & \mathbb{P}(V) \end{tikzcd}$$

$g \mapsto f g f^{-1}$

$Ad_f = d(Ad_f)$

so $\ker(da_\ell) \to f^*\ker(da_{f(\ell)})$

$V \simeq \mathbb{C}^{n+1}$

Example: The tangent bundle of $\mathbb{P}^n = \mathbb{P}(V)$.

Let $a : GL(V) \times \mathbb{P}(V) \to \mathbb{P}(V)$
$$(A, \ell) \longmapsto A(\ell).$$

Then a is a holomorphic map.

Given $\ell \in \mathbb{P}(V)$, let $a_\ell : GL(V) \to \mathbb{P}(V)$ be
$$A \mapsto A(\ell).$$

Claim: $da_\ell|_{id_V} : \underset{\underset{End(V)}{\parallel}}{T_{id} GL(V)} \to T_\ell \mathbb{P}(V)$

is surjective and its kernel is $\{A \in End(V) : A(\ell) \subset \ell\}$. (and given $B \in GL(V)$, $\ker(da_{B(\ell)}|_{id_V}) = B \ker(da_\ell|_{id_V}) B^{-1}$.)

Proof: $GL(V)$ acts transitively on $\mathbb{P}(V)$. Hence it suffices to prove it at one point, say $\ell = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$.
Now work in the open neigh W of $id_V \in GL(V)$,
$W = \{A ; a_{11} \neq 0\}$ and $U_0 \subset \mathbb{P}(V)$. Then
$a_\ell : W \to U_0 \ni (1, *, * \cdots *)$

$$a_\ell(A) = A\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = \frac{1}{a_{11}}\begin{pmatrix}a_{11}\\a_{21}\\ \vdots\\a_{n1}\end{pmatrix}. \quad \text{So}$$

$$da_\ell|_{id_V}(A) = \frac{\partial}{\partial z}\Big|_{z=0}\left(a_\ell(I+zA)\right) = \frac{\partial}{\partial z}\Big|_{z=0}\left(\frac{1}{1+za_{11}}\begin{pmatrix}1+za_{11}\\za_{21}\\ \vdots\\za_{n1}\end{pmatrix}\right) \overset{\text{product rule}}{=}$$

$$= -a_{11}\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} + 1\cdot\begin{pmatrix}a_{11}\\a_{21}\\ \vdots\\a_{n1}\end{pmatrix} = \begin{pmatrix}0\\a_{21}\\ \vdots\\a_{n1}\end{pmatrix}. \quad \text{So } \ker(da_\ell|_{id}) = \begin{pmatrix}* & * & & *\\0 & * & & *\\ \vdots & & & \\0 & * & & *\end{pmatrix} \quad \square$$

The differential $da : T[GL(V) \times \mathbb{P}(V)] \to a^* T\mathbb{P}(V)$ is thus a surjective vector bundle homo.

Over the submanifold $\{id_V\} \times \mathbb{P}(V)$ we get the surjective bundle homo (where $\underline{V} := \mathbb{P}(V) \times V$ is the trivial VB)

$$da : End(\underline{V}) \oplus T\mathbb{P}(V) \to T\mathbb{P}(V).$$

The claim implies that its restriction to the first summand is surjective

$$da_1 : End(\underline{V}) \twoheadrightarrow T\mathbb{P}(V)$$

and its kernel is the subbundle $ker(da_1)$ whose fiber over $\ell \in \mathbb{P}(V)$ is

$$\{A \in End(V) : A(\ell) \subset \ell\}.$$

Consider the short exact seq $0 \to \mathcal{O}_{\mathbb{P}V}(-1) \xrightarrow{\iota} \underline{V} \xrightarrow{\xi} Q \to 0$

We get " " "

$$\begin{tikzcd}
 & & & Hom(\mathcal{O}(-1), \mathcal{O}(-1)) \arrow[d] & \\
0 \arrow[r] & Hom(Q, \underline{V}) \arrow[r] & Hom(\underline{V}, \underline{V}) \arrow[r, two heads, "A \mapsto A \circ \iota"] \arrow[d] \arrow[dr, two heads, "\delta"] & Hom(\mathcal{O}(-1), \underline{V}) \arrow[r] \arrow[d, "\xi \circ (\cdot)"] & 0 \\
 & & Hom(\underline{V}, Q) \arrow[r] & Hom(\mathcal{O}(-1), Q) &
\end{tikzcd}$$

The kernel of da_1 is the same subbundle of $Hom(\underline{V}, \underline{V})$ as the kernel of the SURJECTIVE diagonal homo δ above. Hence

$$T\mathbb{P}(V) \simeq Hom(\underline{V}, \underline{V}) / ker(da_1) = Hom(\underline{V}, \underline{V}) / ker(\delta) \simeq Hom(\mathcal{O}(-1), Q).$$

We get the short exact seq

$$\boxed{0 \to \mathcal{O}_{\mathbb{P}(V)} \to \mathcal{O}(1) \otimes \underline{V} \to T\mathbb{P}(V) \to 0}$$

called the Euler sequence