Huybrechts 1.1.11:

Example of an irreducible analytic set, with a reducible germ:

Let $X = Z(g) \subset \mathbb{C}^2$, where

$g(z_1, z_2) = z_1^3 + z_1^2 - z_2^2$. Then X is analytic and $X = \mathrm{Im}(f)$, where $f: \mathbb{C} \to \mathbb{C}^2$ is the holo function of Exercise 1.1.15.

$\mathbb{C}$ is irreducible, and the image of an irreducible set via a holo map is irreducible. Hence X is irreducible.

Now, the germ of X at $(0,0)$ is reducible.

Indeed, $f(z) = (z^2-1, z(z^2-1))$,

so $f(1) = f(-1) = (0,0)$, and $f'(1) \neq (0,0)$, $f'(-1) \neq (0,0)$.

Let $U_1 \subset \mathbb{C}$ be an open disk centered at 1 of sufficiently small radius. Then $f(U_1)$ is an analytic subset of $\mathbb{C}^2$, by Corollary 1.1.12, since $\mathrm{rk}(J(f)(1)) = 1$ is maximal. Similarly, if $U_{-1} \subset \mathbb{C}$ is an open disk of sufficiently small radius, then $f(U_{-1})$ is an analytic subset.

Finally, the germ of X at $(0,0)$ is the union of the germ X_1 of $f(U_1)$ and the germ X_{-1} of $f(U_{-1})$. So the germ of X

at $(0,0)$ is reducible.

Example of a reducible analytic set, whose induced germs are all irreducible:

Take X to be the union of two disjoint lines $z_1 = 0$ and $z_1 = 1$ in $\mathbb{C}^2$.

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<u>1.1.15</u>: $f: \mathbb{C} \to \mathbb{C}^2$,

$$f(z) = (\underbrace{z^2-1}_{z_1}, \underbrace{z^3-z}_{z_2}) \qquad (z_2/z_1)^2 - 1 = z_1$$

Let $g(z_1, z_2) = z_1^3 + z_1^2 - z_2^2$.

Then $g(f(z)) = (z^6 - 3z^4 + 3z^2 - 1) + (z^4 - 2z^2 + 1) - \underset{(z^6 - 2z^4 + z^2)}{\lambda} = 0$.

Claim $\operatorname{Im}(f) = Z(g)$.

Pf: If $z_1 \neq 0$ and $g(z_1, z_2) = 0$, then

$$0 = z_1 + 1 - \left(\frac{z_2}{z_1}\right)^2 \iff z_1 = \left(\frac{z_2}{z_1}\right)^2 - 1 \text{ and}$$

$$z_2 = \left(\frac{z_2}{z_1}\right)^3 - \left(\frac{z_2}{z_1}\right),$$

so $(z_1, z_2) = f\left(\frac{z_2}{z_1}\right)$.

If $z_1 = 0$ and $g(z_1, z_2) = 0$, then $z_2 = 0$ and so $(z_1, z_2) = (0, 0) = f(1)$. $\square$

Hence $\operatorname{Im}(f)$ is analytic.

Huybrechts Ex 1.1.20:

$U \subset \mathbb{C}^n$ an open subset, $n \geq 2$, and
$f : U \setminus \{z_n = z_{n-1} = 0\} \to \mathbb{C}$ a holo function.
Then there exists a unique holomorphic function $\tilde{f} : U \to \mathbb{C}$ extending f.

Proof: We may assume that U is connected (otherwise, apply separately for each connected component).
Set $\Sigma := U \cap Z(\{z_{n-1}, z_n\})$. Then $U \setminus \Sigma$ is connected and dense in U, by Ex 1.1.8. If $p \in \Sigma$ and $\tilde{f}_1, \tilde{f}_2$ are extensions of f to $(U \setminus \Sigma) \cup B_\varepsilon(p)$, for some polydisk $B_\varepsilon(p)$, then $\tilde{f}_1 = \tilde{f}_2$, by the Identity Theorem since $U \cup B_\varepsilon(p)$ is connected. Hence an extension $\tilde{f}$ is unique. Furthermore, it suffices to extend f to $B_\varepsilon(p)$. So, we may assume $p = 0$, $U = B_\varepsilon(0)$, for some $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$.

Let $\varepsilon' = (\varepsilon_1', \dots, \varepsilon_n')$, where $0 < \varepsilon_1' < \varepsilon_1$, $0 < \varepsilon_2' < \varepsilon_2$, but $\varepsilon_3' = \varepsilon_3, \dots, \varepsilon_n' = \varepsilon_n$. Then $B_{\varepsilon'}(0) \cap \Sigma$ contains $B_\varepsilon(0) \cap \Sigma$ and f is holo on $B_\varepsilon(0) \setminus \Sigma$, so it is holo on $B_\varepsilon(0) \setminus B_{\varepsilon'}(0)$. Now the proof of Prop 1.1.4
page 6 applies to extend f to $B_\varepsilon(0)$, since it used only the strict inequality $\varepsilon_1' < \varepsilon_1$ and $\varepsilon_2' < \varepsilon_2$. $\square$