Problem 2.3.4: The image of $Div(X) \to Pic(X)$ consists of those line bundles admitting non-trivial meromorphic sections.

Proof: Let $\{U_i, \varphi_i : L|_{U_i} \to U_i \times \mathbb{C}\}$ be trivialization of L over an open covering.
If L has a meromorphic non-zero section s, then $\varphi_i(s)$ is a non-zero meromorphic function on U_i and it image in $\mathcal{K}^*_{U_i}/\mathcal{O}^*_{U_i} \simeq Div(U_i)$ is a divisor on U_i. These divisors agree on the overlaps $U_{ij} := U_i \cap U_j$ and so define a global divisor (s) (Remark 2.3.24 (ii)).

The meromorphic functions $\varphi_i(s) \in \mathcal{K}^*(U_i)$ are "defining equations" for the divisor (s) on U_i and so the line bundle $\mathcal{O}((s))$ associated to the divisor (s) in the proof of Cor 2.3.10 is given by the cocycle

$$\left\{ g_{i_0 i_1} := \varphi_{i_0}(s)|_{U_{i_0 i_1}} \cdot \varphi_{i_1}^{-1}(s)|_{U_{i_0 i_1}} \right\}.$$

On the other hand,

$$\varphi_{i_0}(s)|_{U_{i_0 i_1}} = \varphi_{i_0 i_1} \cdot \varphi_{i_1}(s)|_{U_{i_0 i_1}}$$

since s is a global meromorphic section of L. Hence,

$$\varphi_{i_0 i_1} = \varphi_{i_0}(s)|_{U_{i_0 i_1}} \cdot \varphi_{i_1}(s)^{-1}|_{U_{i_0 i_1}} = g_{i_0 i_1}.$$

Hence, $L \simeq \mathcal{O}((s))$ (as claimed in Remark 2.3.24 (iii)).

We conclude that L belong to the image of $Div(X)$,

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Conversely, suppose that $L = \mathcal{O}(D)$, for some divisor $D = \sum a_i [Y_i]$, Y_i irreducible hypersurfaces, $a_i \in \mathbb{Z}$.
Write $D_+ := \sum_{a_i > 0} a_i [Y_i]$, $D_- := \sum_{a_i < 0} a_i [Y_i]$, so that $D = D_+ - D_-$. Let s_+ be a holomorphic section of $\mathcal{O}(D_+)$ with divisor $Z(s_+) = D_+$ and s_- a holo section of $\mathcal{O}(D_-)$ with divisor $Z(s_-)$, as in Prop. 2.3.18 (iii).
Let $\{U_i\}$ be an open covering such that
$\mathcal{O}(D_+)$ has trivializations $\varphi_i : \mathcal{O}(D_+)_{|U_i} \to U_i \times \mathbb{C}$
$\mathcal{O}(D_-)$ has " $\varphi_i' : \mathcal{O}(D_-)_{|U_i} \to U_i \times \mathbb{C}$.
Then $\varphi_i(s_+)/\varphi_i'(s_-) \in K^*(U_i)$ is a meromorphic function on U_i with divisor $D = D_+ - D_-$ on U_i.

$$\varphi_{i_0}(s_+)_{|U_{i_0 i_1}} = \varphi_{i_0 i_1} \cdot \varphi_{i_1}(s_+)_{|U_{i_0 i_1}}$$

$$\varphi_{i_0}'(s_-)_{|U_{i_0 i_1}} = \varphi_{i_0 i_1}' \cdot \varphi_{i_1}'(s_-)_{|U_{i_0 i_1}},$$

so

$$\varphi_{i_0}(s_+)/\varphi_{i_0}'(s_-)_{|U_{i_0 i_1}} = \underbrace{\left(\varphi_{i_0 i_1}/\varphi_{i_0 i_1}'\right)}_{\substack{\text{cocycle of} \\ \mathcal{O}(D_+)\otimes \mathcal{O}(D_-)^* \simeq \mathcal{O}(D_+ - D_-) = \mathcal{O}(D)}}\left(\varphi_{i_1}(s_+)/\varphi_{i_1}'(s_-)_{|U_{i_0 i_1}}\right)$$

Hence, $\left\{\varphi_i(s_+)/\varphi_i'(s_-)\right\}$ glues to a global meromorphic section of $\mathcal{O}(D)$ with divisor D. Hence, $L = \mathcal{O}(D)$ has a global meromorphic section. $\square$