Problem 2.3.10:

Let $C = \varphi_{\mathcal{O}(2)}(\mathbb{P}^1) \subset \mathbb{P}^2$, $\quad \varphi := \varphi_{\mathcal{O}(2)}$

$H^0(\mathbb{P}^1, \mathcal{O}(2)) = \mathrm{Sp}\{z_0^2, z_0 z_1, z_1^2\}$

$H^0(\mathbb{P}^2, \mathcal{O}(1)) = \mathrm{Sp}\{x, y, z\}$. $\varphi(z_0 : z_1) = (z_0^2 : z_0 z_1 : z_1^2)$.

$\varphi^* \mathcal{O}_{\mathbb{P}^2}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(2)$

$\varphi^*(x) = z_0^2$, $\varphi^*(y) = z_0 z_1$, $\varphi^*(z) = z_1^2$.

Then $C = V(y^2 - xz)$ is a smooth conic.

Let $\overset{(a:b:c)}{\overset{\shortparallel}{p}} \in \mathbb{P}^2$, and $\pi : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$, projection from p.

Case I: Assume first that $p \notin C$. Then

$\pi \circ \varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a well defined holomorphic map

$$(\pi \circ \varphi)^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \varphi^*(\underbrace{\pi^* \mathcal{O}_{\mathbb{P}^1}(1)}_{\mathcal{O}_{\mathbb{P}^2 \setminus \{p\}}(1)}) \simeq \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathcal{O}_{\mathbb{P}^1}(2).$$

Furthermore, $\pi \circ \varphi$ is given by the linear subsystem

$\varphi^*(W)$, $\quad W \subset H^0(\mathbb{P}^2, \mathcal{O}(1))$

$W = \{f(x, y, z) : f(p) = 0\} \leftarrow$ 2-dim'l

$\varphi^*(W) = \{f(z_0^2, z_0 z_1, z_1^2) : f \in W\}$

(x, y, z)

If $\eta : \mathbb{C}^3 \to \mathbb{C}^3$ is an isometry w.r.t the quadratic poly $Q(x,y,z) = y^2 - xz$, then $\eta(C) = C$ and $\eta|_C$ is an automorphism of C, so of $\mathbb{P}^1$, so $\eta|_C \in PGL(2, \mathbb{C})$.

Conversely, given $g \in PGL(2)$, the induced action on

$$\mathbb{P}^2 = \mathbb{P}\Big[\underbrace{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))}_{\substack{\| \\ \text{Homog poly of deg 2 in } z_0, z_1}}{}^*\Big]$$

preserves the quadratic form Q' on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^*$ given by $y^2 - xy$, where $\{x, y, z\}$ is the dual basis to the basis $\{z_0^2, z_0 z_1, z_1^2\}$ of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.

We conclude that $PGL(2, \mathbb{C}) = \underset{\substack{\uparrow \\ \text{orthogonal gp} \\ \text{wrt } Q}}{PO}(\mathbb{C}^3, Q)$.

Now, $PO(\mathbb{C}^3, Q)$ acts transitively on complement $\mathbb{P}^2 - \underbrace{V(Q)}_{C}$. Hence, we may normalize the equation choosing $p = (0, 1, 0) \notin C$.

Then $W \subset H^0(\mathbb{P}^2, \mathcal{O}(1))$

$\overset{\|}{\{f(x,y,z) : f(p) = 0\}} = Sp\{x, z\}$.

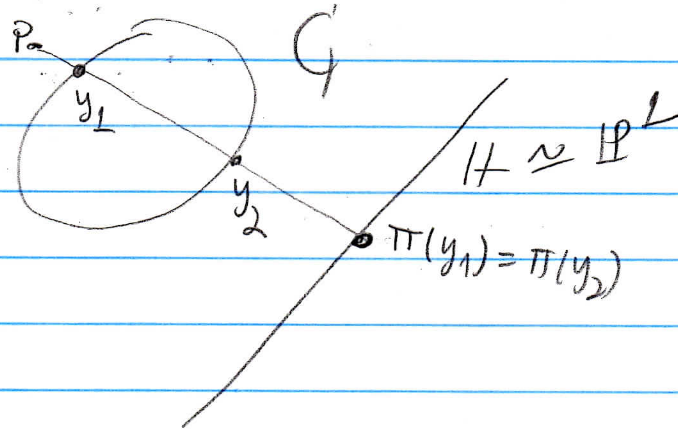
$\ell^*(W) = \{f(z_0^2, z_0 z_1, z_1^2) : f \in W\} = Span\{z_0^2, z_1^2\}$.

So $\pi \circ \ell : \mathbb{P}^1 \to \mathbb{P}^1$ is given by

$$(z_0 : z_1) \longmapsto (z_0^2 : z_1^2).$$

Following is a geometric description:

$\varphi : \mathbb{P}^1 \to \mathcal{C}$ is biholomorphic, by Problem 2.3.5.
Using description (c) of the solution to problem 2.3.7, and Bezout's Theorem (problem 2.3.8) we see that $\pi \circ \varphi$ is a map of degree 2 from $\mathbb{P}^1$ to $\mathbb{P}^1$



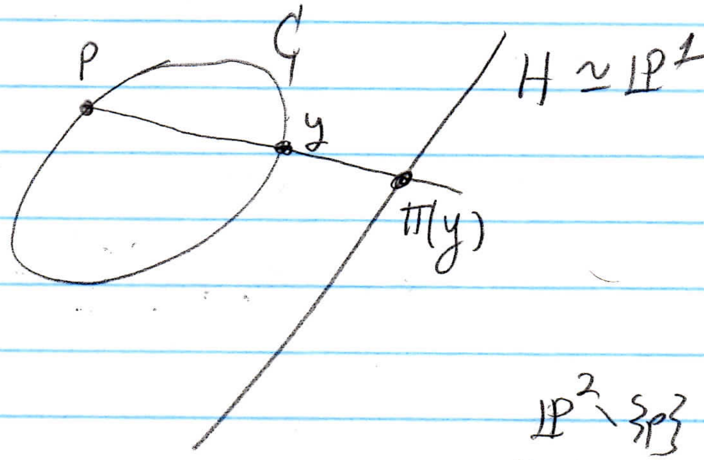
The fiber of $\pi|_{\mathcal{C}} : C \to H$ over $\pi(y_1)$, $y_1 \in \mathcal{C}$, consists of the two points of intersection of the degree 2 curve $\mathcal{C}$, with the line through P and y_1.

Case II: Assume that $P \in \mathcal{C} \simeq \varphi(\mathbb{P}^1)$. Since PGL(2) acts transitively on $\mathbb{P}^1$, we may assume that $P = \varphi(0:1) = (0^2 : 0\cdot 1 : 1^2) = (0:0:1)$,
So $W \subset H^0(\mathbb{P}^2, \mathcal{O}(1))$
$\simeq \{ f(x:y:z) ; f(0:0:1) = 0 \} = Sp\{x, y\}$.

$\varphi^*(W) = \{ z_0^2, z_0 z_1 \}$ $\leftarrow$ The induced map $g = $ "$\pi \circ \varphi$" is given by the linear system

$g : \mathbb{P}^1 \to \mathbb{P}^1$
$g(z_0 : z_1) = (z_0^2 : z_0 z_1) \overset{\text{only the ratio counts}}{=} (z_0 : z_1)$. So g is the identity.

<u>Geometrically</u>: Again $\varphi: \mathbb{P}^1 \to C$ is an isomorphism.



If $p \in C$, then $\pi(\overset{\mathbb{P}^2 \setminus \{p\}}{\overset{\in}{y}})$ is the intersection of the line $l_{p,y}$ through p, y with the line H.
The line $l_{p,y}$ intersect C along two points (Bezout) one of which is p.
So π restricts to C as a one-to-one map onto $\mathbb{P}^1$, in accordance with the above computation using coordinates.