HW 4

Problem 1: (On Dolbeault Coho)

(a) If α is a d-closed (p,q)-form, then

$d\alpha = \underbrace{\partial\alpha}_{(p+1,q)\text{-form}} + \underbrace{\bar\partial\alpha}_{(p,q+1)\text{-form}} = 0$, so $\partial\alpha = 0$ and $\bar\partial\alpha = 0$.

If α is a d-exact $(1,0)$-form, then $\alpha = df = \partial f + \bar\partial f$, for some C^∞-complex valued function f, such that $\bar\partial f = 0$. So f must be holomorphic, $f \in H^0(X, \mathcal{O}_X)$. But X is cpt, so f must be constant. Hence, $df = 0$. We conclude that the space $B_d^{(1,0)}$ of d-exact $(1,0)$-forms is (0).

Hence the inclusion $Z_d^{1,0} \hookrightarrow Z_{\bar\partial}^{1,0}$ induces a well defined homo

$$f^{1,0} : H^{1,0}(X) \to H^{1,0}_{Dol}(X).$$

It is injective, since the space of exact $(1,0)$-forms is (0).

We know that $H^{1,0}_{Dol}(X) \simeq H^0(X, \Omega^1_X)$, by Dolbeault's Theorem. If X is a compact complex torus, T^*X is the trivial vector bundle and so its sheaf of sections Ω^1_X satisfies $H^0(X, \Omega^1_X) = Sp\{dz_1, \ldots, dz_n\}$, where dz_j are a basis for the translation invariant holo 1-forms on the universal cover $\mathbb{C}^n$. $f^{1,0}$ is surjective, since each dz_j is also d-closed. Hence $f^{1,0}$ is an isomorphism.

(b) The inclusion $Z_d^{0,q} \subset Z_{\bar\partial}^{0,q}$ induces a well defined homo

$$\beta^{0,q} : H^{0,q}(X) \to H^{0,q}_{Dol}(X).$$

Let α be a d-closed $(0,q)$-form. Then α is $\bar\partial$-closed as well. If α is d-exact, $\alpha = d\beta$, $\beta = \sum_{r=0}^{q-1} \alpha^{r,q-1-r}$, then

$\alpha = \bar\partial \alpha^{0,q-1}$, so α is also $\bar\partial$-exact. Hence $\beta^{0,q}$ is well-defined.

(c) $X = \mathbb{C}^n/\Lambda$ is an n-dim'l cpt cpx torus. $\pi : \mathbb{C}^n \to X$ the covering map.

(i) We know that $H^{1,0}(X) = \text{Span}\{dz_1, \dots, dz_n\}$ from part (a) and that $H^{0,1}(X) = \overline{H^{1,0}(X)}$ (complex conjugate). Hence $H^{0,1}(X)$ is n-dim'l spanned by $\{d\bar z_1, \dots, d\bar z_n\}$.

(ii) $\beta^{0,1} : H^{0,1}(X) \to H^{0,1}_{Dol}(X)$ is injective.

Let ω be a d-closed 1-form of type $(0,1)$. If $\beta^{0,1}(\omega) = 0$, then $\omega = \bar\partial g$, for a C^∞ complex valued function g, so

$\bar\partial \partial g = -\partial\bar\partial g = -\partial\omega = 0$, since ω is d-closed (hence both $\bar\partial$ and ∂ closed). Hence ∂g is a global holomorphic $(1,0)$-form. So

$\partial g = \sum_{i=1}^{n} t_i \, dz_i$, $t_i \in \mathbb{C}$,

We know from part (c) i that
$\omega = \sum_{i=1}^{n} h_i \, d\bar{z}_i$, $h_i \in \mathbb{C}$.

So $dg = \partial g + \underbrace{\bar{\partial} g}_{\omega} = \sum_{i=1}^{n} f_i \, dz_i + \sum_{i=1}^{n} h_i \, d\bar{z}_i$

So $g \circ \pi = \sum_{i=1}^{n} f_i z_i + \sum_{i=1}^{n} h_i \bar{z}_i + c_1$, for some $c_1 \in \mathbb{C}$.

But $g \circ \pi$ is doubly-periodic. So
$g = c_1$. Hence $\omega = \bar{\partial} g = 0$. $\square$

(iii) We have the short exact exponential seq
$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ and the
long exact

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\text{surj}} \mathbb{C}^* \to 0 \qquad H^0 \text{ row}$$

$$0 \to H^1(X, \mathbb{Z}) \to \underbrace{H^1(X, \mathcal{O}_X)}_{H^{01}(X)} \to \underbrace{H^1(X, \mathcal{O}_X^*)}_{\mathrm{Pic}(X)} \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

Hence, Pic(X) contains the quotient of the n-dimensional complex vector space $H^{01}(X)$ (by Part (c) i) by the rank $2n$ lattice $H^1(X, \mathbb{Z})$. Hence, Pic(X) is non-trivial.