HW 3

Sec 2.1 page 65, #2.1.2:

Let X be a connected compact complex submanifold of $\mathbb{C}^n$. Then the coordinate functions z_i restrict to X as holomorphic functions, which must be constant, by Liouville's Thm (in higher dimension). Hence, the embedding $X \hookrightarrow \mathbb{C}^n$ is a constant map, and X is a point.

Sec. 2.2 page 75 problem 2.2.3:

Given holomorphic vector bundles E, F, we have the canonical isomorphism of vector bundles

$$(*) \qquad Hom(E,F) \xrightarrow{\simeq} E^* \otimes F.$$

In terms of a local frame $\{e_1, -, e_r\}$ of E its sends a section φ of $Hom(E,F)$ to

$$\varphi \longmapsto \sum_{i=1}^{r} e_i^* \otimes \varphi(e_i)$$

where $\{e_1^*, -, e_r^*\}$ is the dual frame of E^*. The isomorphism $(*)$ is independent of the choice of a local frame (and thus glues to a global vector bundle isomorphism, a local frame $\Leftrightarrow$ local trivialization).

Given holomorphic vector bundles V, W, and L we get the isomorphisms

$$(\dagger) \qquad Hom(V \otimes W, L) \underset{\text{by } (*)}{\simeq} (V^* \otimes W^*) \otimes L \underset{\substack{\text{associativity} \\ \text{of tensor} \\ \text{product}}}{\simeq} V^* \otimes (W^* \otimes L) \underset{\text{by } (*)}{\simeq} Hom(V, W^* \otimes L).$$

Let E be a rank r vector bundle. The wedge product bilinear pairing

$$(**) \qquad \bigwedge^{q} E \otimes \bigwedge^{r-q} E \longrightarrow \bigwedge^{r} E$$

is defined in terms of a local frame $\{e_1, -, e_r\}$ by

$$\underbrace{(e_{i_1} \wedge \cdots \wedge e_{i_q}) \otimes (e_{j_1} \wedge \cdots \wedge e_{j_{r-q}})}_{(1)} \mapsto e_{i_1} \wedge \cdots \wedge e_{i_q} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{r-q}}.$$

We know, from linear algebra, that the pairing (**) is canonical, independent of the choice of a local frame (fiber by fiber it is independent of the choice of a basis). Hence, it glues to a global homomorphism of vector bundle (clearly everywhere of rank 1).

Taking $V := \Lambda^{r} E$, $W := \Lambda^{n-r} E$, and $L := \Lambda^{n} E$, in (+), we get that the pairing (**) corresponds to a global holomorphic section h of

$$\mathrm{Hom}\left(\Lambda^{r} E, \Lambda^{n-r} E^{*} \otimes \Lambda^{n} E\right).$$

Fiber by fiber, we know from linear algebra that the pairing (**) is non-degenerate, so that the section h is an isomorphism of the fibers over every point. Hence h is a vector bundle isomorphism.

page 73, 2.2.2 : Let L be a holomorphic line bundle on a (connected) compact complex manifold X, $0 \neq s \in H^0(X, L)$, $h \in H^0(X, L^*)$.
$\neq$
0

Then the locus where the section $s \otimes h$ of $H^0(X, L \otimes L^*)$ vanishes is the union of two proper analytic subset, hence a proper " " . Thus, $s \otimes h$ is a non trivial section of the trivial l.b. $L \otimes L^*$, corresponding to a non zero section of $\mathcal{O}_X$. But $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$, by Liouville's Theorem, so a non-trivial global holo function on X is constant, hence nowhere zero. Hence, s and h are nowhere zero, i.e., global isomorphisms.

Sec 2.2 page 76 problem

2.2.13:

$$\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n/\Lambda =: X$$

Choose an open covering $\{U_i\}_{i \in I}$ of X, such that π restricts to each connected component of $\pi^{-1}(U_i)$ as an isomorphism onto U_i.
Given $i, j \in I$ and connected components $\tilde{U}_i$ of $\pi^{-1}(U_i)$ and $\tilde{U}_j$ of $\pi^{-1}(U_j)$, set
set $\varphi_i := (\pi_i|_{\tilde{U}_i})^{-1} : U_i \longrightarrow \tilde{U}_i$, define $\varphi_j : U_j \to \tilde{U}_j$ similarly. We get

$$\begin{tikzcd}
U_i \arrow[r, "\varphi_i"] & \tilde{U}_i \\
U_i \cap U_j \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\cap"] \arrow[r] \arrow[dr] & \varphi_i(U_i \cap U_j) \arrow[u, phantom, "\cup"] \\
U_j \arrow[r, "\varphi_j"'] & \varphi_j(U_i \cap U_j) \arrow[u, bend right, "\varphi_{ij}"'] \arrow[d, phantom, "\cap"] \\
 & \tilde{U}_j
\end{tikzcd}$$

Now, $\varphi_{ij} = \left(\pi|_{\tilde{U}_i}\big|_{\varphi_i(U_i \cap U_j)}\right)^{-1} \circ \left(\pi|_{\tilde{U}_j}\right)\big|_{\varphi_j(U_i \cap U_j)}$. Hence,

on each connected component C of $\varphi_j(U_i \cap U_j)$ φ_{ij} is given by translation by an element $\gamma_C \in \Lambda$, which depends only on C. Thus, the Jacobian $J(\varphi_{ij}) : \varphi_j(U_i \cap U_j) \longrightarrow GL(n, \mathbb{C})$ is the identity map.

Hence, the cocycle $J(\varphi_{ij})$ defining TX (Def 2.2.14) is trivial, and so TX is the trivial vector bundle. $\square$

The canonical l.b. $K_X \overset{\text{def}}{:=} \overset{n}{\Lambda} T^*X$ is trivial, hence
$$H^0(X, K_X^{\otimes m}) \simeq H^0(X, \mathcal{O}_X) \simeq \mathbb{C}.$$

So $R(X) := \bigoplus_{m \geq 0} H^0(X, K_X^{\otimes m})$ is isomorphic to $\mathbb{C}[t]$, where $\{t\}$ is a basis for $H^0(X, K_X)$. Hence

$$\mathrm{kod}(X) = \underbrace{\mathrm{trdeg}_{\mathbb{C}}(\mathbb{C}[t])}_{1} - 1 = 0.$$

Def 2.2.26

Page 97 problem 2.4.10: Notation:

We let $V=\mathbb{C}^m$, $Gr(r,V)$ = the Grassmannian of r-dimensional subspace of V.

$M_{r,m}$:= subset of $r\times m$ matrices of rank r.

$GL(r,\mathbb{C})$ acts on $M_{r,m}$ and the quotient is $Gr(r,V)$. Let $\pi: M_{r,m}\to Gr(r,V)$ be the (holomorphic) quotient map.

Given an ordered subset $I:=(i_1,\dots,i_r)$ of $\{1,\dots,m\}$, and a matrix $M\in M_{r,m}$, let M_I be the $r\times r$ minor with columns $i_1,\dots,i_r$. Let $\tilde{U}_I\subset M_{r,m}$ be the open subset $=\{M: M_I$ is invertible$\}$.

Set $U_I:=\pi(\tilde{U}_I)\subset Gr(r,V)$. Then $\pi^{-1}(U_I)=\tilde{U}_I$, so that $\{U_I\}$ is an open covering of $Gr(r,V)$.

Let $P_I: M_{r,m}\longrightarrow M_{r,m-r}$ be the map deleting the r columns indexed by I (and here $M_{r,m-r}$ consists of all matrices).

Set $\varphi_I: U_I\longrightarrow M_{r,m-r}\simeq\mathbb{C}^{r\times(m-r)}$,

$$\varphi_I(\pi(M))=P_I\left[M_I^{-1}\cdot M\right].$$

Then φ_I is well defined (check!) and injective onto $M_{r,m-r}$.

Let $\hat{e}_I : M_{r,m} \to M_{r,m}$ be $\hat{e}_I(M) := M_I^{-1} \cdot M$. Clearly, its image $\hat{e}_I(\tilde{M}_{r,m})$ is isomorphic to the image of e_I.

If $M \in \tilde{U}_I \cap \tilde{U}_J$, then the gluing transformation $e_{IJ} : \hat{e}_J(U_{IJ})$ to $\hat{e}_I(U_{IJ})$ is

$$e_{IJ}(\hat{e}_J(\pi(M))) = e_{IJ}(M_J^{-1} M) \stackrel{def}{=} M_I^{-1} M =$$

$$= (M_I^{-1} M_J)\, \hat{e}_J(\pi(M)).$$

Note that $e_{IJ} : \underbrace{\hat{e}_J(U_{IJ})}_{\cong\, M_{r,m-r}} \to \underbrace{\hat{e}_I(U_{IJ})}_{\cong\, M_{r,m-r}}$

is given by left multiplication by values of a well defined holomorphic map

$$\tilde{e}_{IJ} : U_{IJ} \to GL(r, \mathbb{C})$$

$$\tilde{e}_{IJ}(\pi(M)) := M_I^{-1} M_J.$$

$\tilde{e}_{I,J}$ is well defined, since given an element $A \in GL(r, \mathbb{C})$, we have

$$(AM)_I^{-1} (AM)_J = (M_I^{-1} A^{-1})(A M_J) = M_I^{-1} M_J.$$

Now we follow the proof in the $\mathbb{P}V$ case presented in class.

Let
$$a : GL(V) \times Gr(r, V) \to Gr(r, V)$$
$$(A, W) \longmapsto A(W).$$

Then a is a holomorphic map.

Given $W \in Gr(r, V)$, let
$$a_W : GL(V) \to Gr(r, V)$$
$$A \longmapsto A(W).$$

<u>Claim:</u>
$$da_W|_{id_V} : T_{id_V} GL(V) \to T_W Gr(r, V)$$
is surjective and it kernel is
$$\{A \in End(V) : A(W) \subset W\}.$$

<u>Proof:</u> $GL(V)$ acts transitively on $Gr(r,V)$, and given $B \in GL(V)$,
$$\ker(da_{B(W)}|_{id_V}) = B \ker(da_W|_{id_V}) B^{-1}.$$

Hence, it suffices to prove the claim for one subspace, say $V := \mathbb{C}^n$, $W_0 := \Pi(M_0)$, and $M_0 := (I_r|0) \in M_{r,n}$, with $P_{I_0}(M_0) = 0$.
Then $M_0 \in U_{I_0}$, $I_0 = (1, 2, \ldots, r)$

Consider the open subset G_0 of $GL(n, \mathbb{C})$ consisting of A where the top left $r \times r$ block $\hat{A}$ of A is invertible.

Then
$$\varphi_{I_0}(a_{W_0}(A)) = \varphi_{I_0}(A(W_0)) = \varphi_{I_0}(\Pi(M_0 A^T)) = P_{I_0}((\hat{A}^T)^{-1} M_0 A^T)$$

$$\left(da_{W_0}\big|_{id_V}\right)(A) = \frac{\partial}{\partial z}\Big|_{z=0} \left(a_{W_0}(I+zA)\right) =$$

$$= \frac{\partial}{\partial z}\Big|_{z=0} \left(\left(\widehat{(I+zA)}^T\right)^{-1} M_0 \, (I+zA)^T \right) =$$

$$= \underbrace{\frac{\partial}{\partial z}\Big|_{z=0} \left(\left(\widehat{(I+zA)}^T\right)^{-1} \right)}_{=\, -\frac{\partial}{\partial z}\big|_{z=0} \left(\widehat{(I+zA)}^T\right) \,=\, -\hat{A}^T} M_0 + M_0 \underbrace{\frac{\partial}{\partial z}\Big|_{z=0} (I+zA)^T}_{A^T}$$

$$= \underbrace{-\hat{A}^T M_0}_{\left(-\hat{A}^T \mid 0\right)} + \underbrace{M_0 A^T}_{\left(\text{first } r \text{ rows of } A^T\right)} = \begin{pmatrix} 0 & & 0 \\ 0 & & 0 \\ \hline a_{r+1,1} & & a_{r+1,r} \\ \vdots & & \vdots \\ a_{n,1} & & a_{n,r} \end{pmatrix}^T .$$

Surjectivity of $da_{W}\big|_{id_V}$ follows.

Furthermore $\ker\left(da_{W_0}\big|_{id_V}\right) = \left\{ A : a_{ij} = 0 \text{ for } \begin{array}{l} r+1 \le i \le n \\ 1 \le j \le r \end{array} \right\}$

$$= \{A : A W_0 \subset W_0\} . \quad \square$$

The differential $da : T[GL(V)\times Gr(k,V)] \to a^* TGr(k,V)$ is a surjective vector bundle homomorphism, by the claim. Over the submanifold

$$\{id_V\}\times Gr(k,V)$$

we get the surjective vector bundle homo (where $\underline{V} := Gr(k,V)\times V$ is the trivial vector bundle)

$$da : End(\underline{V}) \oplus TGr(k,V) \to TGr(k,V),$$

and the claim implies that its restriction to the first summand is surjective as well

$$da_1 : End(\underline{V}) \twoheadrightarrow TGr(k,V)$$

and its kernel is the subbundle, whose fibers over $W\in Gr(k,V)$ is

$$\{A\in End(V) : A(W)\subset W\}.$$

Consider the short exact sequence

$$0\to S \xrightarrow{e} \underline{V} \xrightarrow{q} Q \to 0$$

of the tautological subbundle S, whose fibers over $W\in Gr(k,V)$ is W. We get the commutative diagram:

$$\begin{tikzcd}
 & A \arrow[r, mapsto] & A\circ e & \\
0 \arrow[r] & Hom(Q,\underline{V}) \arrow[r] & Hom(\underline{V},\underline{V}) \arrow[r, two heads, "e_*"'] \arrow[d] \arrow[dr, "\delta"] & Hom(S,\underline{V}) \arrow[r] \arrow[d, "q^*"] & 0 \\
 & & Hom(\underline{V},Q) \arrow[r] & Hom(S,Q) &
\end{tikzcd}$$

Both e_* and q^* are surjective, hence $\delta := q^*\circ e_*$ is surjective and $\delta(A) = q\circ A\circ e$. Hence, $ker(\delta) = ker(da_1)$. Thus,

$$TGr(k,V) \simeq End(\underline{V})/ker(da_1) \simeq Hom(\underline{V},\underline{V})/ker(\delta) \simeq Hom(S,Q). \qquad \square$$