Hint for problem 1.2.1

1. Given a compatible almost complex structure (CACS), construct an orthonormal basis $\{v_1, v_2, v_3, v_4\}$, such that $I(v_1) = v_2$ and $I(v_3) = v_4$. Conclude that any two CACS are conjugate under the orthogonal group $O(V)$, and the conjugacy class has at most two connected components.

2. Show that the stabilizer of a CACS I in $SO(V)$ is four dimensional, by computing its tangent space at the identity as a subspace of the Lie algebra $so(4, \mathbb{R})$ of skew-symmetric matrices. Conclude that each $SO(V)$ orbit is two dimensional.

3. Let $\mathbb{H} := \{a1+bI+cJ+dK \ : \ a, b, c, d \in \mathbb{R}\}$ be the quaternions with inner product given by $(q_1, q_2) := q_1\bar{q}_2$, where $I^2 = J^2 = K^2 = -1$, $IJ = K$, and conjugation is given by
$$\overline{a + bI + cJ + dK} = a - bI - cJ - dK.$$
Let Σ be the set of unit purely imaginary quaternions
$$\Sigma := \{aI + bJ + cK \ : \ a^2 + b^2 + c^2 = 1\}.$$
Given $q \in \Sigma$, let $L_q : \mathbb{H} \to \mathbb{H}$ be left multiplication by q and let $R_q : \mathbb{H} \to \mathbb{H}$ be right multiplication by q. Show that both L_q and R_q are compatible almost complex structures and that the two embeddings of Σ in $O(\mathbb{H})$ are disjoint.

Remark: The *Pfaffian* $Pf(M)$ of a $2n \times 2n$ skew-symmetric matrix M is an invariant polynomial with respect to the $SO(2n)$ conjugation action satisfying $Pf(M)^2 = \det(M)$. For 4×4 matrices it is given by
$$Pf \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + dc.$$
The matrix of a $CACS$ with respect to an orthonormal basis is skew-symmetric and so its Pfaffian is an invariant of each connected component of the $O(V)$-conjugacy class.