1. For which values of $\lambda \in \mathbb{C}$ is the projective cubic curve $C_\lambda$

$$x^3 + y^3 + z^3 + 3\lambda xyz = 0$$

in $\mathbb{P}^2$ singular? Show that each singular curve $C_\lambda$ in the family is a union of three lines.

2. For which values of $\lambda \in \mathbb{C}$ is the projective cubic curve $C_\lambda$

$$y^2z - x(x - z)(x - \lambda z) = 0$$

in $\mathbb{P}^2$ singular? Show that all curves in the family are irreducible. Show that if $C_\lambda$ is singular, than it has a single singular point $p_0$, and $C_\lambda \setminus \{p_0\}$ is isomorphic to $\mathbb{C} \setminus \{0\}$.

3. Show that the projective cubic plane curve $C$, given by $zy^2 - x^3$, is irreducible and has precisely one singular point $p_0$. Show that $C \setminus \{p_0\}$ is isomorphic to $\mathbb{C}$.

4. Miranda, section II.2 page 38: B, F.

5. Let $w_1, w_2$, be two complex numbers, which are linearly independent over $\mathbb{R}$, and let $L$ be the lattice $\mathbb{Z}w_1 + \mathbb{Z}w_2$. The Weierstrass $\mathcal{P}$-function is

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{w \in L, w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right).$$

It is shown in standard complex analysis textbooks (Lang, Ahlfors, etc ...) that $\mathcal{P}$ is absolutely convergent uniformly on compact subsets of $\mathbb{C} \setminus L$ and $\mathcal{P}$ is $L$-periodic, and so descends to a meromorphic function on the complex torus $X := \mathbb{C}/L$ ($L$-periodicity is obvious for the derivative $\mathcal{P}'$). Note, that $\mathcal{P}$ is even $\mathcal{P}(z) = \mathcal{P}(-z)$. It is characterized as the unique $L$-periodic function, holomorphic on $\mathbb{C} \setminus L$, whose Laurent series centered at 0,

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

satisfies $a_{-2} = 1$, and $a_n = 0$, for all $n \leq 0$, except $a_{-2}$.

(a) Show that the derivative $\mathcal{P}'$ has a simple zero at each half period

$$\{u : u \in (\mathbb{C} \setminus L) \text{ and } 2u \in L\}$$

(corresponding to three points in $X$). Hint: Use the fact that $\mathcal{P}$ is even and Lemma 3.14 in the text.

(b) Assume now that $w_1 = 1$, $w_2 = \tau$, and $u \in \{\frac{1}{2}, \frac{\tau}{2}, \frac{1 + \tau}{2}\}$. Let $\theta^{(u)}(z)$ be the function defined in page 34 of the text. Show that the difference $\mathcal{P}(z) - \mathcal{P}(u)$ is a constant multiple of

$$\frac{\theta^{(u)}(z)\theta^{(-u)}(z)}{[\theta^{(0)}(z)]^2}.$$