1. Let $P(z)$ and $Q(z)$ be polynomials and suppose that $\text{deg} \ Q \geq \text{deg} \ P + 2$. Show that
\[ \sum \text{Res}_\zeta (P/Q) = 0, \]
where the sum runs over all singularities of the rational function $P/Q$. Do this problem in two ways: (1) Directly, without considering the residue at $\infty$. (2) As a special case of problem 3.

2. Lang Problem 37 page 190: Let $f$ be analytic on $\mathbb{C}$ with exception of a finite number of isolated singularities $\{z_1, \ldots, z_n\}$. Define the residue at infinity
\[ \text{Res}_\infty f(z) dz := -\frac{1}{2\pi i} \int_{|z|=R} f(z) dz \]
for $R > \max\{|z_1|, \ldots, |z_n|\}$.
(a) Show that $\text{Res}_\infty f(z) dz$ is independent of $R$.
(b) Show that the sum of the residues of $f$ in the extended complex plane $\mathbb{CP}^1$ is equal to zero. (This result is often referred to as the Residue Theorem.)

3. Lang Problem 38 page 190 (Cauchy’s Residue Formula on the Riemann Sphere).

4. Basic Exam, September 1998 Problem 5: Compute $\int_C \frac{z^4 e^{1/z}}{1 - z^4} dz$ where $C$ denotes the circle $\{ |z| = 2 \}$ transversed counterclockwise. Hint: Use the residue at infinity (problem 3) to save computations.

5. Basic Exam, January 2000 Problem 7: Let $f$ be a one-to-one holomorphic map from a region $D_1$ onto a region $D_2$. Suppose that $D_1$ contains the closure of the disk $\Delta := \{ |z - z_0| < \rho \}$. Prove that for $w \in f(\Delta)$ the inverse function $f^{-1}(w)$ is given by
\[ f^{-1}(w) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f'(z)}{f(z) - w} \cdot z dz \]

6. Let $P(z)$ be a polynomial of degree $d$ and assume that the roots $\zeta_1, \ldots, \zeta_d$ of $P$ are simple. Show that for $R$ sufficiently large:
\[ \int_{|z|=R} \frac{z^k P'(z)}{P(z)} dz = \sum_{i=1}^{d} \zeta_i^k. \]

7. Lang page 189 Problem 29: Let $U$ be a connected open set, and let $D$ be an open disk whose closure is contained in $U$. Let $f$ be analytic on $U$ and not constant. Assume that the absolute value $|f|$ is constant on the boundary of $D$. Prove that $f$ has at least one zero in $D$. Hint: Consider $g(z) := f(z) - f(z_0)$ with $z_0 \in D$.

8. Basic Exam January 2000 Problem 9: Prove that the equation
\[ 20 \frac{z^3}{z^2 + 4} = e^z \]
has 3 roots in the unit disk $\{ |z| < 1 \}$. 


9. Basic Exam, August 99 Problem 4: Let \( \lambda \) be a real number larger that 1. Show that the equation \( \lambda - z - e^{-z} = 0 \) has exactly one solution in the half plane \( \{ z : \text{Re}(z) > 0 \} \). Moreover, the solution is real.

10. Basic Exam, January 99 problem 3:

   (a) Determine the number of zeroes of \( z^5 - 2z^2 + z + 1 \) in the disk \( \{ z : |z| < 10 \} \).

   (b) Compute the integral \( \int_{\{z:|z|<10\}} \frac{3z^4 + 1}{z^5 - 2z^2 + z + 1} \, dz \).

11. (Recommended Problem, you need not handed it in) Lang Page 191 Problem 40 (modified): Let \( a, b \in \mathbb{C} \) with \(|a| \) and \(|b| < R \). Let \( C_R \) be the circle of radius \( R \).

   a) Show that there is an analytic branch of \( \sqrt{(z-a)(z-b)} \) defined in \( \{ z : |z| > R \} \) and such that \( \frac{z}{\sqrt{(z-a)(z-b)}} \) is analytic with value 1 at \( \infty \).

   b) Evaluate \( \int_{C_R} \frac{zd\,z}{\sqrt{(z-a)(z-b)}} \).

   c) Evaluate \( \int_{C_R} \frac{d\,z}{\sqrt{(z-a)(z-b)}} \).