1. Ahlfors, page 136 Problem 6: If \( \gamma \) is a path, piecewise of type \( C^1 \), contained in the open unit disc \( D \), then the integral
\[
\int_{\gamma} \frac{|dz|}{1 - |z|^2}
\]
is called the non-euclidean length or hyperbolic length of \( \gamma \). Let \( f : D \to D \) be an analytic function from the disc into itself. Show that \( f \) maps every \( \gamma \) on a path with smaller or equal non-euclidean length. Deduce that a linear fractional transformation from \( D \) onto itself preserves non-euclidean lengths.

2. Ahlfors, page 136 Problem 7: (Modified) It can be shown, that the path of smallest non-euclidean length, joining the origin 0 to a point \( z \in D \), is the straight line segment between them.
   
   (a) Use this fact to show that the path of smallest non-euclidean length, that joins two given points in the unit disk, is the piece of the circle \( C \) which is orthogonal to the unit circle \( \partial D \). The shortest non-euclidean length is called the non-euclidean distance.
   
   (b) Show that the non-euclidean distance between \( z_1 \) and \( z_2 \) is
\[
\frac{1}{2} \log \frac{1 + \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}{1 - \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}
\]

3. (a) Show that single valued analytic branches of \( f(z) = z^\alpha \), \( \alpha \in \mathbb{C} \), and \( f(z) = z^z \) can be defined in any simply connected region, which does not contain the origin.
   
   (b) (Ahlfors, problem 5 page 148) Show that a single vlaued analytic branch of \( \sqrt{1 - z^2} \) can be defined in any region such that the points 1 and \(-1\) are in the same connected component of the complement. What are the possible values of \( \int \frac{dz}{\sqrt{1 - z^2}} \) over a closed curve in the region?

4. Laurent Serries: Lang page 164: 8, 12, 13

5. Problem 5 from the basic exam of August 99: Consider the Laurent series
\[
\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ which is valid in the annulus } \frac{\pi}{2} < |z| < \frac{3\pi}{2}. \text{ Find the coefficients } a_n \text{ with index } -\infty < n \leq -1. \text{ Hint: Use integration.}
\]

6. Isolated Singularities: Lang page 170: 1a,c,e, 4

7. Find the value of the integral \( \int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} \, dz \) for:
   
   (a) \( C \) the circle \( |z - 2| = 2 \).
   
   (b) \( C' \) the circle \( |z| = 4 \).
8. Suppose that \( f(z) = \frac{g(z)}{h(z)} \), \( g(z_0) \neq 0 \) and \( h(z) \) has a zero of order 2 at \( z_0 \). Prove that

\[
Res_{z_0} f = \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h'''(z_0)}{3(h''(z_0))^2} - 2g(z_0)h''(z_0)
\]

9. Compute the integral of the following functions over the circle \(|z| = 2|:

(a) \( f(z) = \frac{1}{(z - 3)(1 + 2z)^2(1 - 3z)^3} \)

(b) \( f(z) = \frac{z^2}{1 - e^{z/4}} \)

(c) \( f(z) = \frac{\cos(1/z)}{1 + z^4} \)