AN APPENDIX TO: THE BEAUVILLE-BOGOMOLOV CLASS AS A CHARACTERISTIC CLASS

EYAL MARKMAN

Contents

1. Introduction 1
2. Singular moduli spaces and their deformations 1
3. O’Grady’s results on the structure of the normal cone 3
4. Extracting the data \( \{ \mathbb{P}V, \mathbb{P}V^* \} \) from the normal cone 3

References 4

1. Introduction

Let \( X \) be any compact Kähler manifold deformation equivalent to the Hilbert scheme \( S^{[n]} \) of length \( n \) subschemes on a K3 surface \( S \), \( n \geq 2 \). Such a manifold will be called below of \( K3^{[n]} \)-type. Let \( \Delta \subset X \times X \) be the diagonal. The paper “The Beauville-Bogomolov class as a characteristic class” carries out a construction of a \( \mathbb{P}^{2n-3} \)-bundle \( \mathbb{P}V \) over \( \tilde{\Sigma} := [X \times X] \setminus \Delta \), which corresponds to a slope-stable twisted reflexive sheaf over \( X \times X \), with monodromy-invariant characteristic classes (see [M]). We first constructed \( \mathbb{P}V \) when \( X \) is a moduli space of sheaves. We then used Verbitsky’s theory of hyperholomorphic sheaves in order to deform the construction of \( \mathbb{P}V \) to every \( X \) as above.

In this appendix we provide another geometric interpretation of the above construction. Let \( \Sigma \) be the complement of the diagonal in the second symmetric product of \( X \). Then \( \tilde{\Sigma} \) is the universal cover of \( \Sigma \). Now \( \Sigma \) is expected to be a stratum of the singular locus of a \( \mathbb{Q} \)-factorial compact holomorphic symplectic variety \( Y \), at least when \( X \) is a sufficiently “small” deformation of \( S^{[n]} \) (Conjecture 2.2). We reconstruct the projective bundle \( \mathbb{P}V \) over \( \tilde{\Sigma} \) from the geometry of the projectivized normal cone of \( \Sigma \) in \( Y \). Conjecture 2.2, if true, would thus provide an alternative construction of the pair \( (X, \mathbb{P}V) \) constructed in [M], for \( X \) in the local Kuranishi deformation space of \( S^{[n]} \).

2. Singular moduli spaces and their deformations

Let \( S \) be a projective K3 surface with a cyclic Picard group generated by an ample line-bundle \( H \) of degree 2. Let \( K_{\text{top}}S \) be the topological \( K \) group
of $S$ and denote by $v \in K_{\text{top}}S$ the class of the ideal sheaf of a length $n$ subsheaf of $S$, so that the moduli space $M_H(v)$, of $H$ stable coherent sheaves of class $v$, is simply the Hilbert scheme $S^{[n]}$. Assume that $n \geq 2$. The equivalence class (also known as the S-equivalence class), of an $H$-semistable sheaf, is the isomorphism class of the associated graded sheaf, with respect to the Harder-Narasimhan filtration. Let $M_H(2v)$ be the moduli space of equivalence classes of $H$-semistable sheaves over $S$, with class $2v$. If $n \geq 3$, then $M_H(2v)$ is an irreducible locally factorial singular projective symplectic variety with terminal singularities, which does not admit a crepant resolution. [KLS]. When $n = 2$ the moduli space $M_H(2v)$ is Q-factorial and it does admit a crepant resolution [O’G, LS]. The singularities of $M_H(2v)$ determine a stratification

$$M_H(2v) \supset M(2v)_{\text{sing}} \supset M(v),$$

$M(2v)_{\text{sing}}$ is isomorphic to the second symmetric product $\text{Sym}^2 M(v)$, and $M(v) \hookrightarrow \text{Sym}^2 M(v)$ is the diagonal embedding. A point in $M(2v)_{\text{sing}}$ corresponds to the S-equivalence class of the direct sum $I_{Z_1} \oplus I_{Z_2}$ of two ideal sheaves, with $Z_j$, $j = 1, 2$, a length $n$ subscheme of $S$.

Let $\mathcal{Y} \to \text{Def}(M_H(2v))$ be the semi-universal family over the local Kuranishi deformation space of $M_H(2v)$. Namikawa studied the deformation theory of Q-factorial projective symplectic varieties with terminal singularities [Nam1, Nam2, Nam3]. His results imply that $\text{Def}(M_H(2v))$ is smooth ([Nam1, Theorem 2.5]). Furthermore, the semi-universal family $\mathcal{Y}$ is locally trivial [Nam3]. So deformations of $M_H(2v)$ remain singular and the deformation $p : \mathcal{Y}_{\text{sing}} \to \text{Def}(M_H(2v))$ of their singular loci is locally trivial over $\text{Def}(M_H(2v))$, for $n \geq 3$. Local triviality means that given a point $y \in \mathcal{Y}_{\text{sing}}$, there exists an analytic open neighborhoods $U$ of $y$ in $\mathcal{Y}_{\text{sing}}$, $U_1$ of $p(y)$ in $\text{Def}(M_H(2v))$, and $U_2$ of $y$ in the fiber $\mathcal{Y}_{\text{sing}}$ over $p(y)$, and an isomorphism $U \cong U_1 \times U_2$, which conjugates $p$ to the projection onto $U_1$.

**Corollary 2.1.** The fiber $Y$, over a generic point of $\text{Def}(M_H(2v))$, is singular, with a stratification

$$Y \supset Y_{\text{sing}} \supset X,$$

where the reduced singular locus $Y_{\text{sing}}$ is isomorphic to $\text{Sym}^2(X)$, $X$ is smooth of $K^3[n]$-type, and the inclusion $X \subset Y_{\text{sing}}$ is the diagonal embedding.

**Proof.** Simply use Namikawa’s local triviality twice. Once for the semi-universal family $\mathcal{Y} \to \text{Def}(M_H(2v))$, in order to conclude the flatness of $p : \mathcal{Y}_{\text{sing}} \to \text{Def}(M_H(2v))$, and once to conclude the local triviality of $p$. □

Corollary 2.1 gives rise to a natural morphism of local deformation spaces

$$\text{Def}(M_H(2v)) \longrightarrow \text{Def}(M_H(v)),$$

sending $Y$ to the smallest stratum of its singular locus. When $n \geq 3$, both moduli spaces are smooth and 23-dimensional. Recall that $M_H(v) = S^{[n]}$. 
Conjecture 2.2. The morphism (2.2) is surjective, for a generic polarized K3 surface \((S, H)\).

Note that it would suffice to prove that the differential of the morphism (2.2) is invertible, a calculation we have not carried out.

3. O’Grady’s results on the structure of the normal cone

Set
\[ \Sigma := \mathcal{M}(2v)_{\text{sing}} \setminus \mathcal{M}(v), \]
\[ \widetilde{\Sigma} := \left[ \mathcal{M}(v) \times \mathcal{M}(v) \right] \setminus \Delta_{\mathcal{M}(v)} \].

Then \( \widetilde{\Sigma} \to \Sigma \) is the universal cover and we let \( \tau : \widetilde{\Sigma} \to \widetilde{\Sigma} \) denotes its Galois involution. Let \( \mathcal{E} \) be the universal ideal sheaf over \( S \times \mathcal{M}_H(v) \). Denote by \( \pi_{ij} \) the projection from \( \mathcal{M}_H(v) \times S \times \mathcal{M}_H(v) \) onto the product of the \( i \)-th and \( j \)-th factors. Denote by \( V \) the restriction to \( \widetilde{\Sigma} \) of the sheaf \( \mathcal{E} \times \mathcal{E} \)
\begin{equation}
\mathcal{E} \times \mathcal{E} (\pi_{12}^* \mathcal{E}, \pi_{23}^* \mathcal{E}) .
\end{equation}

Then \( V \) is a locally free sheaf of rank \( 2n - 2 \). Let \( \tilde{q} \in \text{Sym}^2[V \oplus V^*] \) be the symmetric bilinear form \( \tilde{q}(x, y) = y(x) \) and \( C_{\Sigma} \subset \mathbb{P}V \times \mathbb{P}V^* \) the subscheme defined by \( \tilde{q} = 0 \). A fiber of \( C_{\Sigma} \), over \( \sigma \in \tilde{\Sigma} \), is the incidence divisor
\begin{equation}
Q \subset \left[ \mathbb{P}^{2n-3} \times (\mathbb{P}^{2n-3})^* \right].
\end{equation}

When \( n = 2 \), \( \mathbb{P}V \) is a \( \mathbb{P}^1 \)-bundle, hence self-dual, and \( C_{\Sigma} \) is the graph of the isomorphism \( \mathbb{P}V \cong \mathbb{P}V^* \).

The pullback \( \tau^* V \) is isomorphic to \( V^* \). Thus, the vector bundle \( V \oplus V^* \), the quadratic form \( \tilde{q} \), the fiber product \( \mathbb{P}V \times \mathbb{P}V^* \), and its subvariety \( C_{\Sigma} \), descend to a vector bundle over \( \Sigma \) with a quadratic form \( q \), a \( \mathbb{P}^{2n-3} \times (\mathbb{P}^{2n-3})^* \)-bundle \( \mathcal{P} \) over \( \Sigma \), and a subvariety
\[ C_{\Sigma} \subset \mathcal{P} . \]

Proposition 3.1. ([O’G], Proposition 1.4.1 and Theorem 1.2.1) \( C_{\Sigma} \) is isomorphic to the projectivized normal cone of \( \Sigma \) in \( \mathcal{M}_H(2v) \).

4. Extracting the data \( \{ \mathbb{P}V, \mathbb{P}V^* \} \) from the normal cone

Assume that conjecture 2.2 holds. Let \( X \) be an irreducible holomorphic symplectic manifold, parametrized by a point \( [X] \) in \( \text{Def}(S^{[n]}) \) in the image of a point \( [Y] \) in \( \text{Def}(\mathcal{M}_H(2v)) \) via the morphism (2.2). Let \( \Sigma := [\text{Sym}^2 X] \setminus \Delta \) and \( \tilde{\Sigma} := [X \times X] \setminus \Delta \) be the complements of the diagonals. Then \( \Sigma \) is a stratum in \( Y_{\text{sing}} \) and the projectivized normal cone \( C_{\Sigma} \), of \( \Sigma \) in \( Y \), is a \( \mathcal{Q} \)-bundle, where \( \mathcal{Q} \) is the incidence divisor (3.2), by Corollary 2.1.

If \( n = 2 \), the pullback of \( C_{\Sigma} \) to \( \tilde{\Sigma} \) is the \( \mathbb{P}^1 \)-bundle we are looking for\(^1\). Assume \( n \geq 3 \). Then the relative Picard sheaf \( \text{Pic}(C_{\Sigma}/\Sigma) \) is a \( \mathbb{Z} \oplus \mathbb{Z} \) local

---

\(^1\)In the case \( n = 2 \) one need not consider the whole of \( \text{Def}(\mathcal{M}_H(2v)) \), but rather the divisor along which the fiber \( Y \) of the semi-universal family remains singular.
system, and it has a canonical double section, whose value at a point \( \sigma \in \Sigma \) is the pair of two generators of the effective cone of the fiber \( Q \) of \( C_\Sigma \) over \( \sigma \). The double section is connected, as it is connected in the case \( \Sigma = M(2v)_{\text{sing}} \setminus M(v) \). Hence, the double section is isomorphic to the universal cover \( \tilde{\Sigma} \) of \( \Sigma \). Let \( C_{\tilde{\Sigma}} \) be the fiber product \( C_\Sigma \times_{\Sigma} \tilde{\Sigma} \). Then \( \text{Pic}(C_{\tilde{\Sigma}}/\tilde{\Sigma}) \) has an unordered pair of two sections \( \{L_1, L_2\} \) (labeled by a choice of identification of \( \tilde{\Sigma} \) with the double section), such that \( L_i \) “restricts” to the fiber \( Q \) of \( C_{\tilde{\Sigma}} \to \tilde{\Sigma} \) as the line bundle \( \mathcal{O}_Q(1,0) \) or \( \mathcal{O}_Q(0,1) \) of the incidence divisor (3.2). Each \( L_i \) determines a \( \mathbb{P}^{2n-3} \)-bundle \( P_i \) over \( \tilde{\Sigma} \) (of linear systems along the fibers), and a morphism \( \eta_i : C_{\tilde{\Sigma}} \to \mathbb{P}_i^* \). The morphisms \( \eta_1, \eta_2 \) are the two rulings of \( C_{\tilde{\Sigma}} \). The embedding

\[
(\eta_1, \eta_2) : C_{\tilde{\Sigma}} \to \mathbb{P}_1^* \times \mathbb{P}_2^*
\]

determines an isomorphism \( \mathbb{P}_1^* \cong \mathbb{P}_2^* \). When \( X = M(v) \cong S[n] \) and \( Y = \mathcal{M}_H(2v) \), the dual pair \( \{\mathbb{P}_1, \mathbb{P}_2\} \) is precisely the pair \( \{\mathbb{P}V, \mathbb{P}V^*\} \), where \( V \) is given in (3.1).

References


