Algebraic Geometry Homework Assignment 8, Fall 2007
Due Tuesday, November 27.

The field $k$ below is assumed algebraically closed.

1. (Hartshorne, Problem I.5.6, Blowing up curve singularities) Let $Y = V(f)$ be an affine plane curve and $P = (a, b)$ a point of $\mathbb{A}^2$. Write $f = f_{\mu} + f_{\mu+1} + \ldots + f_d$, where $f_i$ is a homogeneous polynomial of degree $i$ in $(x-a)$ and $(y-b)$, and $f_{\mu} \neq 0$. Recall that the multiplicity of $P$ on $Y$ is $\mu$. If $\mu > 0$, the tangent directions are cut out by the linear factors of $f_{\mu}$. A double point is a point of multiplicity 2. We define a node (also called an ordinary double point) to be a double point with distinct tangent directions. Denote by $\varphi: \tilde{Y} \to Y$ the morphism of blowing-up $P \in Y$.

(a) Let $Y$ be the cuspidal curve $V(y^2 - x^3)$ or the nodal curve $V(x^6 + y^6 - xy)$ from Homework 7 question 6. Show that the curve $\tilde{Y}$, obtained by blowing up $Y$ at the point $O := (0, 0)$, is non-singular. Note: The term cusp is defined in Exercise I.5.14 part d in Hartshorne. It is characterised also as a double point planar singularity $p \in Y$, such that $\varphi^{-1}(P)$ consists of a single point $\tilde{P} \in \tilde{Y}$ and $\tilde{Y}$ is non-singular at $\tilde{P}$.

(b) Let $P$ be a node on a plane curve $Y$. Show that $\varphi^{-1}(P)$ consists of two distinct non-singular points on $\tilde{Y}$. We say that “blowing-up $P$ resolves the singularity at $P$”.

(c) Let $P = (0, 0)$ be the tacnode of $Y = V(x^4 + y^4 - x^2)$ from Homework 7 question 6. Show that $\varphi^{-1}(P)$ is a node. Using 1b we see that the tacnode can be resolved by two successive blowing-up.

(d) Let $Y$ be the plane curve $V(y^3 - x^5)$, which has a higher order cusp at $O$. Show that $O$ is a triple point; that blowing-up $O$ gives rise to a double point, and that one further blowing-up resolves the singularity.

2. (Hartshorne, Problem I.5.7) Let $Y \subset \mathbb{P}^2$ be a non-singular plane curve of degree $> 1$, defined by the equation $f(x, y, z) = 0$. Let $X \subset \mathbb{A}^3$ be the affine variety defined by $f$ (this is the cone over $Y$). Let $P = (0, 0, 0)$ be the vertex of the cone and $\varphi: \tilde{X} \to X$ the blowing-up of $X$ at $P$.

(a) Show that $P$ is the only singular point of $X$.

(b) Show that $\tilde{X}$ is non-singular (cover it with open affine subsets).

(c) Show that $\varphi^{-1}(P)$ is isomorphic to $Y$.

3. (Hartshorne, Problem I.5.8)

(a) (Euler’s Lemma) Let $f$ be a homogeneous polynomial of degree $m$ in the variables $x_0, \ldots, x_n$. Show that $\sum_{i=0}^n x_i \left( \frac{\partial f}{\partial x_i} \right) = m \cdot f$. Conclude, in particular, that if $\text{char}(k) = 0$ or does not divide $m$, and the partials $\frac{\partial f}{\partial x_i}, 0 \leq i \leq n$, all vanish at a point $P \in \mathbb{P}^n$, then $P$ belongs to $V(f)$. 

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(b) Let $Y \subset \mathbb{P}^n$ be a projective variety of dimension $r$. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogeneous polynomials which generate $I(Y)$. Let $P = (a_0, \ldots, a_n)$ be a point of $Y$. Show that $P$ is a non-singular point of $Y$, if and only if the rank of the matrix \( \left( \frac{\partial f_i}{\partial x_j}(a_0, \ldots, a_n) \right) \) is $n - r$. Hint:

i. Show that this rank is independent of the homogeneous coordinates chosen for $P$.

ii. Pass to an open affine $U_i \subset \mathbb{P}^n$ containing $P$ and use the affine Jacobian matrix.

iii. Use part 3a.

4. (a) Let $f, g \in k[x_0, x_1, x_2]$ be homogeneous polynomial of positive degree. Assume that both $f$ and $g$ vanish at the point $P \in \mathbb{P}^2$. Set $h := fg$. Prove that $\frac{\partial h}{\partial x_i}(P) = 0$, for $0 \leq i \leq 2$.

(b) (Hartshorne, Problem I.5.9) Let $f \in k[x_0, x_1, x_2]$ be a homogeneous polynomial, $Y := V(f) \subset \mathbb{P}^2$ the algebraic set defined by $f$, and suppose that for every $P \in Y$ we have $\frac{\partial f}{\partial x_i}(P) \neq 0$, for some $i$. Show that $f$ is irreducible, and hence that $Y$ is a non-singular variety). Hint: Use problem 8 in Homework 5.

(c) (Hartshorne, Problem I.5.5) For every degree $d > 0$, and for every $p = 0$ or a prime number, give the equation of a non-singular curve of degree $d$ in $\mathbb{P}^2$ over a field $k$ of characteristic $p$.

5. (Hartshorne, Problem I.5.12 part c) Assume that $\text{char}(k) \neq 2$, and let $Q := V(f) \subset \mathbb{P}^n$, where $f(x_0, \ldots, x_n) = x_0^2 + \cdots + x_r^2$, $2 \leq r \leq n$. Recall that any irreducible homogeneous polynomial of degree 2 is equivalent to such an $f$, after a suitable linear change of variables (Homework 3 Question 3). Show that $Q$ is non-singular, if $r = n$, and the singular locus $\text{Sing}(Q)$ is a $\mathbb{P}^{n-r-1}$ linearly embedded in $\mathbb{P}^n$, if $r < n$.

6. (Hartshorne, Problem I.5.15 part b, modified) Let $S := k[X, Y, Z]$, and denote by $\mathcal{H}(d, 2) := \mathbb{P}S_d$ the parameter variety of all curves of degree $d$ in $\mathbb{P}^2$, as in Homework 7 Question 5 and Homework 5 Question 5. Note that $\mathcal{H}(d, 2)$ is isomorphic to $\mathbb{P}^N$, $N = \binom{d + 2}{2} - 1$. Show that the irreducible non-singular curves of degree $d$ correspond to the points of a non-empty Zariski open subset of $\mathcal{H}(d, 2)$.

Hint: Let $F(X, Y, X, T_0, \ldots, T_N)$ be the defining bi-homogeneous equation of the universal curve $C$ in $\mathbb{P}^2 \times \mathcal{H}(d, 2)$, as in HW5 Q5. Consider the bi-homogeneous polynomials $\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}$ and use the completeness of $\mathbb{P}^2$ (Mumford, section I.9 Theorem 1), together with questions 3 and 4. Note: The results of questions 4 and 6 generalize for hypersurfaces in $\mathbb{P}^n$, using the same argument.

7. Blowing-up points of projective varieties: We defined in class the blowing-up $\varphi : \widetilde{Y} \to Y$ of a point $P$ in any variety $Y$. Here you will show that if $Y$ is projective then $\widetilde{Y}$ is projective.

(a) Let $n \geq 1$, $P$ the point $(1, 0, \ldots, 0)$ in $\mathbb{P}^n$, and $\pi : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$ the projection, given by $\pi(a_0, \ldots, a_n) = (a_1, \ldots, a_n)$, as in Homework 4 question
2. Let $x_0, \ldots, x_n$ be the homogeneous coordinates of $\mathbb{P}^n$ and $y_1, \ldots, y_n$ those of $\mathbb{P}^{n-1}$. Prove the following statements (reduce to the affine case).

i. The closure $X$ of the graph of $\pi$ in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ is equal to $V(J)$, where $J$ is the bi-homogeneous ideal generated by $x_iy_j - x_jy_i$, $1 \leq i, j \leq n$.

ii. The restriction $\varphi : X \to \mathbb{P}^n$ of the first projection restricts to an isomorphism $X \setminus \varphi^{-1}(P) \to \mathbb{P}^n \setminus \{P\}$.

iii. The second projection $\psi : X \to \mathbb{P}^{n-1}$ restricts to an isomorphism from $\varphi^{-1}(P)$ onto $\mathbb{P}^{n-1}$.

iv. $X$ is a closed and non-singular subvariety (irreducible) of $\mathbb{P}^n \times \mathbb{P}^{n-1}$.

Definition: Given a subvariety $Y$ of $\mathbb{P}^n$ containing the point $P$, let $\bar{Y}$ be the closure in $X$ of $\varphi^{-1}(Y \setminus \{P\})$. Denote by $\varphi : \bar{Y} \to Y$ also the restriction of the morphism $\varphi$. Then $\bar{Y}$ is the blowing-up of $Y$ at $P$.

(b) Find bihomogeneous equations for the blow-up $\bar{C} \subset \mathbb{P}^2 \times \mathbb{P}^1$ of the point $(1,0,0)$ on $C := V(y^2x - z^2(x + z)) \subset \mathbb{P}^2$. Show that $\bar{C}$ is a non-singular projective curve and the second projection $\psi : \bar{C} \to \mathbb{P}^1$ is an isomorphism.

8. (a) Let $X$ be a compact Riemann surface and $f$ a non-zero element of its function field $K(X)$. Prove that $\text{ord}_P(f) = 0$, for all but finitely many points of $X$. Define the degree $\deg(f)$ of $f$ as the sum $\sum_{P \in X : \text{ord}_P(f) > 0} \text{ord}_P(f)$ of all positive valuations of $f$. Show that $\deg : K(X) \setminus \{0\} \to \mathbb{Z}$ is a homomorphism from the multiplicative group of non-zero rational functions to the integers.

(b) Automorphisms of $\mathbb{P}^1$ (Hartshorne, section I.6 problem 6.6 page 47). Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a fractional linear transformation of $\mathbb{P}^1$ by sending $x \mapsto (ax + b)/(cx + d)$, for $a, b, c, d \in \mathbb{K}$, $ad - bc \neq 0$.

i. Show that a fractional linear transformation induces an automorphism of $\mathbb{P}^1$. We denote the group of all these fractional linear transformations by $PGL(2)$.

ii. Let $\text{Aut}(\mathbb{P}^1)$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\text{Aut}(\mathbb{P}^1)$ is isomorphic to $\text{Aut}(k(x))$, the group of all automorphisms of $k(x)$ as a $k$-algebra.

iii. Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $PGL(2) \to \text{Aut}(\mathbb{P}^1)$ is an isomorphism. See Example II.7.1.1 in Hartshorne for the generalization to the case of $\mathbb{P}^n$.

Hint: Note that the homomorphism $\deg : k(x) \setminus \{0\} \to \mathbb{Z}$ is $\text{Aut}(\mathbb{P}^1)$-invariant.

9. Hartshorne, section I.6 problem 6.7 page 47. Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbb{A}^1$. Show that if $\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}$, then $r = s$. Is the converse true?

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1In the next homework assignment, $\deg(f)$ will be shown to be equal to the degree of the morphism $X \to \mathbb{P}^1$ induced by $f$. 
