1. Prove that the set $C := \{(t, t^2, t^3) : t \in k\}$ is an algebraic subset of $\mathbb{A}^3$.

2. Let $I_1 = (x^2 + y, x)$ and $I_2 = (y^2 x^2 + x^2 + y^3 + y + xy, yx^2 + y^2 + x)$. Show the equality of the algebraic subsets $V(I_1) = V(I_2)$ in $\mathbb{A}^2(\mathbb{Q})$, over the fields $\mathbb{Q}$ of rational numbers.

3. Let $k$ be an algebraically closed field, $X$ an algebraic subset of $\mathbb{A}^n(k)$, and $P$ a point of $\mathbb{A}^n(k)$, which is not in $X$. Show that there is a polynomial $F$ in $k[x_1, \ldots, x_n]$, such that $F(Q) = 0$, for all $Q \in X$, but $F(P) = 1$.

4. (a) If $I_1$ and $I_2$ are ideals of some commutative ring $R$, show that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.

(b) If $I_1$ and $I_2$ are radical ideals, show that $I_1 \cap I_2$ is a radical ideal.

5. Let $k$ be algebraically closed, and $X \subset \mathbb{A}^3(k)$ the union of the $x_1$-axis and the point $(1, 1, 1)$. Find generators for $I(X)$.

6. Let $k$ be a field of characteristic $\neq 2$. Prove that there are three points $a, b, c \in \mathbb{A}^2(k)$, such that

$$\sqrt{(x^2 - 2xy + y^2, y^3 - y)} = m_a \cap m_b \cap m_c,$$

where $m_a$ is the maximal ideal of the point $a$, etc...

Hint: Interpret both sides geometrically.

7. Let $k$ be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. Prove that $V(I)$ is a single point, if and only if $\sqrt{I}$ is a maximal ideal.

8. Let $k$ be an algebraically closed field.

(a) Show that the polynomial $y^2 - x(x - 1)(x - \lambda)$ is irreducible, for every $\lambda \in k$.

Hint: Use Eisenstein’s Criterion, or otherwise.
(b) Show also that the polynomial \(y^2 - x^3\) is irreducible.

9. Definitions

i Let \(X \subset \mathbb{A}^n(k)\) be an affine algebraic subset. The \textit{affine coordinate ring} of \(X\) is the ring \(R := k[x_1, \ldots, x_n]/I(X)\).

ii Let \(A\) be an integral domain and \(K\) its fraction field. Recall that the \textit{integral closure} of \(A\) is the subring \(\overline{A}\) of \(K\), consisting of all elements of \(K\), which are integral over \(A\). \(A\) is said to be \textit{integrally closed}, if \(A = \overline{A}\).

(a) Let \(k\) be an algebraically closed field, \(R\) the coordinate ring of the affine cubic plane curve \(V(Y^2 - X^3)\), and \(K\) the fraction field of \(R\). Prove that \(R\) is not integrally closed, i.e., find an element of \(K\), which is integral over \(R\), but does not belong to \(R\).

Notational suggestion: Denote the images of \(X\) and \(Y\) in \(R\) by \(x\), \(y\).

(b) Repeat part 9a, but with the nodal cubic curve \(V(Y^2 - X^2(X-1))\).

Note: We will later see, that an affine algebraic curve is smooth and connected (to be defined), if and only if its coordinate ring is integrally closed.